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# ON THE COMPUTATION OF ALL THE EQUILIBRIUM POINTS IN HAMILTONIAN SYSTEMS WITH THREE DEGREES OF FREEDOM

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**Abstract.** In dynamical system theory the determination of the equilibrium points often requires the solution of systems of transcendental equations, whose exact number of solutions cannot be found analytically. In this paper, topological degree theory (especially the Kronecker–Picard integral) is implemented to obtain the exact number of these solutions, within a given region. These results are studied and applied to the accurate computation of the total number of equilibrium points of Hamiltonian systems with three degrees of freedom.

## 1. Introduction

The equilibrium points of dynamical systems are usually solutions of simultaneous nonlinear algebraic and/or transcendental equations and their total number can seldom be determined analytically. One can find the total number of equilibrium points within a given region using the topological degree theory [1] and, in particular, Picard's extension [7]. The value of the topological degree can be obtained by Kronecker's integral [1] as well as by any degree computation method. An efficient one is Kearfott's method [5].

Applying Picard's theory we are able to calculate with certainty the total number of the equilibrium points of a dynamical system. In the present contribution this method is applied to a Hamiltonian system with three degrees of freedom, namely the three-dipole problem [2].

## 2. The Topological Degree for the Computation of the Total Number of Roots

Consider the equation $F_n(x) = \mathcal{O}_n$ ($\mathcal{O}_n = (0,\ldots,0)$ denotes the origin of $\mathbb{R}^n$), where $F_n = (f_1,\ldots,f_n)\colon \mathcal{D}_n \subset \mathbb{R}^n \to \mathbb{R}^n$ is a function defined and twice continuously differentiable in a bounded domain $\mathcal{D}_n$ of $\mathbb{R}^n$ with boundary $\mathrm{b}(\mathcal{D}_n)$, whose zeros are simple and not located on $\mathrm{b}(\mathcal{D}_n)$. Then the *topological degree of* $F_n$ *at* $\mathcal{O}_n$ *relative to* $\mathcal{D}_n$ is defined by the following sum [3]:

$$\deg[F_n,\mathcal{D}_n,\mathcal{O}_n] = \textstyle\sum_{x\in F_n^{-1}(\mathcal{O}_n)} \operatorname{sgn} J_{F_n}(x),$$

where $J_{F_n}$ stands for the Jacobian determinant and sgn denotes the sign function.

Since $\deg[F_n,\mathcal{D}_n,\ \mathcal{O}_n]$ is equal to the number of simple roots of $F_n(x)$ which give positive Jacobian, minus the number of simple roots which give negative Jacobian, the total number $\mathcal{N}^r$ of these roots can be obtained by the value of $\deg[F_n,\mathcal{D}_n,\ \mathcal{O}_n]$, if the Jacobian possesses the same sign at these roots. So, Picard [7, 1, 4] has extended the function $F_n$ and the domain $\mathcal{D}_n$ as follows: $F_{n+1} = (f_1,\ldots,f_n,f_{n+1})\colon \mathcal{D}_{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, where $f_{n+1} = y\, J_{F_n}$ and $\mathcal{D}_{n+1}$ is the direct product of $\mathcal{D}_n$ with an arbitrary interval of the real



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$y$–axis containing the point $y = 0$. Then the system:

$$f_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, \dots, n, \qquad y\, J_{F_n}(x_1, x_2, \dots, x_n) = 0, \tag{1}$$

and the equation $F_n(x) = \mathcal{O}_n$ have the same simple solutions, provided $y = 0$. Since the Jacobian of (1) is equal to $(J_{F_n}(x))^2$, it is always positive and, therefore, the total number $\mathcal{N}^r$ of zeros of $F_n(x) = \mathcal{O}_n$ can be given by the following relation: $\mathcal{N}^r = \deg[F_{n+1}, \mathcal{D}_{n+1}, \mathcal{O}_{n+1}]$.

Here, we consider a Hamiltonian system with three degrees of freedom and we compute the number of its equilibrium points in various regions. To this end, we consider the respective transcendental equation $F_3(X) = \mathcal{O}_3$ where $F_3 = (f_1, f_2, f_3)$: $\mathcal{D}_3 \subset \mathbb{R}^3 \to \mathbb{R}^3$, whose zeros are the equilibrium points of our dynamical system. According to Picard's extension we define the function $F_4 = (f_1, f_2, f_3, f_4)$: $\mathcal{D}_4 \subset \mathbb{R}^4 \to \mathbb{R}^4$ and the corresponding system:

$$\begin{gathered} f_1(x_1, x_2, x_3) = 0, \quad f_2(x_1, x_2, x_3) = 0, \quad f_3(x_1, x_2, x_3) = 0, \\ f_4(x_1, x_2, x_3, x_4) = x_4\, J_{F_3}(x_1, x_2, x_3) = 0. \end{gathered} \tag{2}$$

Here $J_{F_3}$ denotes the Jacobian determinant of $F_3$ and $\mathcal{D}_4$ is a rectangular parallelepiped in $\mathbb{R}^4$ given by $\mathcal{D}_3 \times [-\gamma, \gamma]$, with $\gamma$ an arbitrary positive constant. Provided the roots are simple, which means $J_{F_3}(x_1, x_2, x_3) \neq 0$ for $(x_1, x_2, x_3) \in F_3^{-1}(\mathcal{O}_3)$, it is easily seen that System (2) has, in $\mathcal{D}_4$, the same solutions with $F_3(X) = \mathcal{O}_3$. The total number $\mathcal{N}^r$ of simple zeros of $F_3$ in $\mathcal{D}_3$ is given by the value of $\deg[F_4, \mathcal{D}_4, \mathcal{O}_4]$.

### 2.1. KRONECKER INTEGRAL APPROACH

The topological degree can be represented by the Kronecker integral as follows:

$$\deg[F_n, \mathcal{D}_n, \mathcal{O}_n] = \frac{1}{\Omega_n} \iint \cdots \int_{b(\mathcal{D}_n)} \frac{\sum_{i=1}^{n} A_i\, dx_1 \dots dx_{i-1}\, dx_{i+1} \dots dx_n}{(f_1^2 + f_2^2 + \dots + f_n^2)^{n/2}}, \tag{3}$$

where $A_i$ define the following determinants:

$$A_i = (-1)^{n(i-1)} \begin{vmatrix} f_1 & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{i-1}} & \frac{\partial f_1}{\partial x_{i+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ f_2 & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_{i-1}} & \frac{\partial f_2}{\partial x_{i+1}} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_n & \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_{i-1}} & \frac{\partial f_n}{\partial x_{i+1}} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix},$$

and $\Omega_n := 2\pi^{n/2}/\Gamma(n/2)$ denotes the surface of a hypersphere in $\mathbb{R}^n$ with radius unity.

For the computation of the topological degree of $F_4$ we apply Kronecker integral (3) for $n = 4$.

$$\deg[F_4, \mathcal{D}_4, \mathcal{O}_4] = \frac{\Gamma(2)}{2\pi^2} \iiint_{b(\mathcal{D}_4)} \frac{\sum_{i=1}^{4} A_i\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_4}{({f_1}^2 + {f_2}^2 + {f_3}^2 + {f_4}^2)^2}.$$

The determinants $A_1$, $A_2$, $A_3$ and $A_4$, defined earlier, assume the form:

$$A_1 = J_{F_3} \begin{vmatrix} f_1 & \partial_2 f_1 & \partial_3 f_1 \\ f_2 & \partial_2 f_2 & \partial_3 f_2 \\ f_3 & \partial_2 f_3 & \partial_3 f_3 \end{vmatrix}, \quad A_2 = J_{F_3} \begin{vmatrix} f_1 & \partial_1 f_1 & \partial_3 f_1 \\ f_2 & \partial_1 f_2 & \partial_3 f_2 \\ f_3 & \partial_1 f_3 & \partial_3 f_3 \end{vmatrix}, \quad A_3 = J_{F_3} \begin{vmatrix} f_1 & \partial_1 f_1 & \partial_2 f_1 \\ f_2 & \partial_1 f_2 & \partial_2 f_2 \\ f_3 & \partial_1 f_3 & \partial_2 f_3 \end{vmatrix},$$

$$A_4 = x_4 \begin{vmatrix} f_1 & \partial_1 f_1 & \partial_2 f_1 & \partial_3 f_1 \\ f_2 & \partial_1 f_2 & \partial_2 f_2 & \partial_3 f_2 \\ f_3 & \partial_1 f_3 & \partial_2 f_3 & \partial_3 f_3 \\ J_{F_3} & \partial_1 J_{F_3} & \partial_2 J_{F_3} & \partial_3 J_{F_3} \end{vmatrix},$$

where $\partial_i$ denotes differentiation with respect to $x_i$.

For the calculation of the above integral one may use the accurate but time-consuming scheme given by O'Neil and Thomas [6], which is based on Gauss-Legendre quadrature. We have also got rough but true approximations of $\mathcal{N}^r$ by successively applying Weddle's method of integration combined with Romberg's extrapolation to the zero for the calculation of the corresponding volume integrals.

### 2.2. KEARFOTT'S APPROACH

Kearfott's method [5] for the computation of the topological degree is briefly described below. Suppose that $S^{n-1} = \langle x_1, x_2, \ldots, x_n \rangle$ is an $(n-1)$–simplex [5] in $\mathbb{R}^n$ and assume $F_n = (f_1, f_2, \ldots, f_n) : S^{n-1} \to \mathbb{R}^n$ is continuous. Then the *range simplex associated with $S^{n-1}$ and $F_n$*, denoted by $\mathcal{R}(S^{n-1}, F_n)$, is an $n \times n$ matrix with elements $\varrho_{ij}$, $1 \leq i, j \leq n$ given by: $\varrho_{ij} = 1$, if $f_j(x_i) \geq 0$, and $\varrho_{ij} = -1$, if $f_j(x_i) < 0$.

$\mathcal{R}(S^{n-1}, F_n)$ is called *usable* if one of the following conditions holds:

*a*) the elements $\varrho_{ij}$ of $\mathcal{R}(S^{n-1}, F_n)$, are: $\varrho_{ij} = 1$, if $i \geq j$, and $\varrho_{ij} = -1$, if $j = i + 1$.

*b*) $\mathcal{R}(S^{n-1}, F_n)$ can be put into this form by a permutation of its rows.

When $\mathcal{R}(S^{n-1}, F_n)$ is usable, then the parity $\mathrm{Par}\big(\mathcal{R}(S^{n-1}, F_n)\big)$ is defined to be 1, if the number of the permutations of the rows required to put $\mathcal{R}(S^{n-1}, F_n)$ into the form in a) is even. If this number is odd then $\mathrm{Par}\big(\mathcal{R}(S^{n-1}, F_n)\big)$ is defined to be $-1$. For all other cases, we set $\mathrm{Par}\big(\mathcal{R}(S^{n-1}, F_n)\big) = 0$. Suppose that $\mathcal{P}^n$ is an $n$–dimensional polyhedron for some $n \geq 2$ and that $\{S_i^{n-1}\}_{i=1}^m$ is a finite set of $(n-1)$–simplexes with disjoint interiors such that $\sum_{i=1}^m S_i^{n-1} = b(\mathcal{P}^n)$; then, under some assumptions regarding $S_i^{n-1}$, the value of the topological degree of $F_n$ at $\mathcal{O}_n$ relative to $\mathcal{P}^n$ can be obtained by the following relation: $\deg[F_n, \mathcal{P}^n, \mathcal{O}_n] = \sum_{i=1}^m \mathrm{Par}\big(\mathcal{R}(S_i^{n-1}, F_n)\big)$. Kearfott's degree computation method is very efficient and has the advantage that it requires only the signs of function values to be correct.

## 3. Numerical applications

We consider three celestial bodies with a spherical mass distribution and possessing magnetic dipole fields. They are positioned on the vertices of an equilateral triangle and are moving around their common centre of mass on circular orbits. Their electromagnetic moments are supposed to be $\mathbf{M}_1 = (0,0,1)$, $\mathbf{M}_2 = (0,0,\lambda)$ and $\mathbf{M}_3 = (0,0,\kappa)$, where $\kappa, \lambda$ are parameters. A charged particle with negligible mass is moving in their vicinity. The equilibrium positions of this particle are obtained by solving the following system of nonlinear equations [2]:

$$
\begin{aligned}
x_1 &+ \frac{x_1+\mu-1}{r_1^3} + \frac{\lambda(x_1+\mu)}{r_2^3} + \frac{\kappa(x_1+\mu-0.5)}{r_3^3} \\
&- 3x_2 \left( \frac{x_2(x_1+\mu-1)}{r_1^5} + \frac{\lambda x_2(x_1+\mu)}{r_2^5} + \frac{\kappa(x_1+\mu-0.5)(x_2-\sqrt{3}/2)}{r_3^5} \right) \\
&+ x_1 \left( \frac{1}{r_1^3} - \frac{3(x_1+\mu-1)^2}{r_1^5} \frac{\lambda}{r_2^3} - \frac{3\lambda(x_1+\mu)^2}{r_2^5} + \frac{\kappa}{r_3^3} - \frac{3\kappa(x_1+\mu-0.5)^2}{r_3^5} \right) = 0, \\
x_2 &+ \frac{x_2}{r_1^3} + \frac{\lambda x_2}{r_2^3} + \frac{\kappa(x_2-\sqrt{3}/2)}{r_3^3} - 3x_1 \left( \frac{x_2(x_1+\mu-1)}{r_1^5} + \frac{\lambda x_2(x_1+\mu)}{r_2^5} + \frac{\kappa(x_1+\mu-0.5)(x_2-\sqrt{3}/2)}{r_3^5} \right) \\
&- x_2 \left( \frac{1}{r_1^3} + \frac{3x_2^2}{r_1^5} - \frac{\lambda}{r_2^3} + \frac{3\lambda x_2^2}{r_2^5} - \frac{\kappa}{r_3^3} - \frac{3\kappa(x_2-\sqrt{3}/2)^2}{r_3^5} \right) = 0, \\
x_3 &\left[ x_1 \left( \frac{x_2}{r_1^5} - \frac{\lambda x_2}{r_2^5} + \frac{\kappa(x_2-\sqrt{3}/2)}{r_3^5} \right) + x_2 \left( \frac{x_1+\mu-1}{r_1^5} - \frac{\lambda(x+\mu)}{r_2^5} + \frac{\kappa(x+\mu-0.5)}{r_3^5} \right) \right] = 0.
\end{aligned}
$$

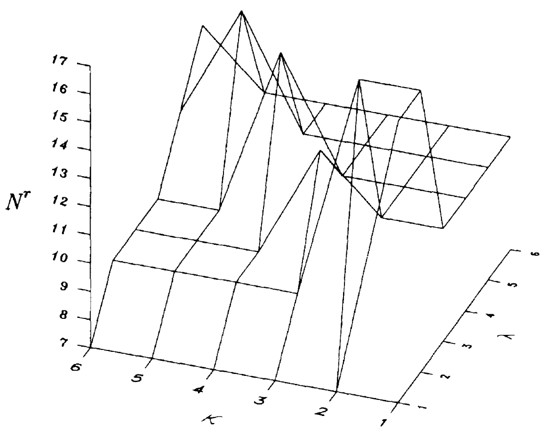


*Figure 1.* The variation of the number of equilibrium points, $\mathcal{N}^r$, versus the parameters $\lambda$ and $\kappa$.

TABLE 1. Number of equilibrium points for $\mu = 0.2$

| $\lambda$ / $\kappa$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 17 | 7 | 7 | 7 | 7 | 7 |
| 2 | 17 | 17 | 9 | 9 | 9 | 9 |
| 3 | 11 | 11 | 13 | 9 | 9 | 9 |
| 4 | 11 | 11 | 11 | 15 | 9 | 9 |
| 5 | 11 | 11 | 11 | 11 | 15 | 11 |
| 6 | 11 | 11 | 11 | 11 | 11 | 13 |

where $\mu$ is a mass parameter of the primaries and

$$r_1 = \sqrt{(x_1 + \mu - 1)^2 + x_2^2 + x_3^2}, \qquad r_2 = \sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2},$$

$$r_3 = \sqrt{(x_1 + \mu - 0.5)^2 + (x_2 - \sqrt{3}/2)^2 + x_3^2}.$$

We have studied the number of equilibrium points for $\mu = 0.2$ and for values of the parameters $\lambda, \kappa = 1, 2, \ldots, 6$. Both Kronecker integral and Kearfott's method have been successfully applied and given the same results. These results are exhibited in Table 1 and verify those given in [2]. Figure 1 reveals that the number of equilibrium points $\mathcal{N}^r$ greatly varies with $\kappa$ and $\lambda$ and that it is maximum when these parameters are equal.

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