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A METHOD FOR COMPUTING FAMILIES OF PERIODIC ORBITS BASED ON UNCONSTRAINED OPTIMIZATION

M.N. VRAHATIS, O. RAGOS AND G.S. ANDROULAKIS
University of Patras, Department of Mathematics, GR-26110 Patras, Greece

Abstract. A new efficient method is presented for the numerical computation of families of periodic orbits of systems with three or more degrees of freedom. This method is based on a well-known procedure using the variational equations as well as on unconstrained optimization techniques. This combination accelerates the convergence of previous schemes. The new composite method has been implemented here on an example of interest to Celestial Mechanics.

1. Introduction

In the study of dynamical systems the computation of periodic solutions is of great importance. Especially for non-integrable such systems it is impossible to obtain complete information regarding any orbit unless it is asymptotic, periodic, or almost periodic. Besides, Poincaré's conjecture suggests that, since the periodic solutions lie "densely" in the phase space, they can be used as reference orbits.

In many cases these orbits form families, namely groups of solutions whose coordinates in phase space vary continuously while their properties change smoothly.

The technique most commonly used for the computation of such families is the construction of a first order predictor-corrector scheme (we name this method LT) based on the solution of a linear system of equations and involving the variational equations of the problem. In this paper we propose a modification of this technique based on optimization (we name this method OT), whose efficiency is demonstrated on a known family of the photogravitational three-body problem.

2. Description of the problem – The techniques

Let us consider a dynamical system of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{f} = (f_1, f_2, \ldots, f_n) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and t is the independent variable. Any solution $\mathbf{x}$ of this system is periodic of period T if it satisfies the condition:

$$\mathbf{x}(\mathbf{x}_0, t = 0) = \mathbf{x}(\mathbf{x}_0, t = T), \tag{1}$$

where $\mathbf{x}_0$ is the initial point of the orbit at $t = 0$.

C. Simó (ed.), *Hamiltonian Systems with Three or More Degrees of Freedom*

Suppose that a periodic solution is known and that it belongs to a specified family. Then, one can compute the whole family calculating successive orbits of it by predicting an approximation of a nearby periodic solution and then by correcting this prediction.

In the sequel, we shall describe two techniques which can be utilized for this purpose by demonstrating their use in the computation of an already known family of the photogravitational restricted circular three-body problem. This dynamical system describes the motion of a small particle under the influence of the gravitation and the radiation forces due to the light emission from the members of a binary star. A classical approach for the behaviour of the small body is given by the following equations:

$$\ddot{x}_1 - 2\dot{x}_2 = \frac{\partial U}{\partial x_1}, \qquad \ddot{x}_2 + 2\dot{x}_1 = \frac{\partial U}{\partial x_2}, \qquad \ddot{x}_3 = \frac{\partial U}{\partial x_3}, \tag{2}$$

where

$$U = \frac{1}{2}(x_1^2 + x_2^2) + \frac{q_1(1-\mu)}{\sqrt{(x_1+\mu)^2 + x_2^2 + x_3^2}} + \frac{q_2\mu}{\sqrt{(x_1+\mu-1)^2 + x_2^2 + x_3^2}},$$

and $1-\mu$, μ express the masses of the two main bodies and q_1, q_2 are parameters representing the relations between the gravitation attraction and the radiation pressure of each one of them ($0 < \mu \le 0.5$, $q_i \le 1$, i=1,2) [4]. This system, under suitable conditions, has, among others, two equilibrium points, named L_6 and L_7, which lie on the Ox_1x_3-plane and are symmetrical with respect to the Ox_1x_2-plane. For some combinations of the parameters of the problem these points are stable and then two families of periodic orbits are emanating from each one of them, named L_6^1, L_6^2 and L_7^1, L_7^2, respectively [5]. The orbits of these families are symmetrical with respect to the Ox_1x_3-plane. In this paper we deal with the family L_6^1.

A first approximation of initial conditions of a small periodic orbit belonging to L_6^1 in the vicinity of L_6 can be obtained by the use of first or second order expansions of Eqs (2) around this point. Then this approximation must be corrected to a desired accuracy to give the exact periodic solution. After this, other members of the family have to be estimated and corrected successively.

A. The method LT constructs, for this purpose, a linear predictor-corrector algorithm based on Taylor expansions around points satisfying the periodicity condition (1) [2, 6]. A suitable algorithm for our example is the following:

a. Find proper modifications $\delta\mathbf{x}_0 = (\delta x_{10}, 0, \delta x_{30}, 0, \delta\dot{x}_{20}, 0)$ and δT of the initial conditions $\mathbf{x}_0 = (x_{10}, x_{20} = 0, x_{30}, \dot{x}_{10} = 0, \dot{x}_{20}, \dot{x}_{30} = 0)$ and the period T of an already known orbit by considering a constant deviation d to one of the δ's and, then, predicting the others from the solution of the system:

$$\begin{aligned}
g_1 &= \frac{\partial x_2}{\partial x_{10}}\delta x_{10} + \frac{\partial x_2}{\partial x_{30}}\delta x_{30} + \frac{\partial x_2}{\partial \dot{x}_{20}}\delta \dot{x}_{20} + \frac{\partial x_2}{\partial t}\delta T = 0,\\
g_2 &= \frac{\partial \dot{x}_1}{\partial x_{10}}\delta x_{10} + \frac{\partial \dot{x}_1}{\partial x_{30}}\delta x_{30} + \frac{\partial \dot{x}_1}{\partial \dot{x}_{20}}\delta \dot{x}_{20} + \frac{\partial \dot{x}_1}{\partial t}\delta T = 0,\\
g_3 &= \frac{\partial \dot{x}_3}{\partial x_{10}}\delta x_{10} + \frac{\partial \dot{x}_3}{\partial x_{30}}\delta x_{30} + \frac{\partial \dot{x}_3}{\partial \dot{x}_{20}}\delta \dot{x}_{20} + \frac{\partial \dot{x}_3}{\partial t}\delta T = 0,
\end{aligned} \tag{3}$$

so that the orbit with initial point $\mathbf{x}_0 + \delta\mathbf{x}_0$ will be approximately periodic of period $T + \delta T$.

b. If the prediction is not satisfactory then alter the initial conditions and the period to obtain a better approximation: apply corrections $\delta\mathbf{x}_0 = (\delta x_{10}, 0, \delta x_{30}, 0, \delta\dot{x}_{20}, 0)$ considering one of the δ's equal to zero and, then, finding the rest of them by solving the equations:

$$
\begin{aligned}
h_1 &= x_2 + \frac{\partial x_2}{\partial x_{10}}\delta x_{10} + \frac{\partial x_2}{\partial x_{30}}\delta x_{30} + \frac{\partial x_2}{\partial \dot{x}_{20}}\delta\dot{x}_{20} + \frac{\partial x_2}{\partial t}\delta T = 0, \\
h_2 &= \dot{x}_1 + \frac{\partial \dot{x}_1}{\partial x_{10}}\delta x_{10} + \frac{\partial \dot{x}_1}{\partial x_{30}}\delta x_{30} + \frac{\partial \dot{x}_1}{\partial \dot{x}_{20}}\delta\dot{x}_{20} + \frac{\partial \dot{x}_1}{\partial t}\delta T = 0, \\
h_3 &= \dot{x}_3 + \frac{\partial \dot{x}_3}{\partial x_{10}}\delta x_{10} + \frac{\partial \dot{x}_3}{\partial x_{30}}\delta x_{30} + \frac{\partial \dot{x}_3}{\partial \dot{x}_{20}}\delta\dot{x}_{20} + \frac{\partial \dot{x}_3}{\partial t}\delta T = 0,
\end{aligned}
\tag{4}
$$

so that the point $\mathbf{x}_0 + \delta\mathbf{x}_0$ will be a better estimation of a periodic orbit of the family of period $T + \delta T$.

The above mentioned coordinates of the orbits and the partial derivatives are evaluated at $t = T$.

B. We now give the method OT. Let us assume that the family is described by a function of the initial conditions and the period of its orbits, $f(\mathbf{x}_0^p, T) = 0$. This means that all these orbits are zeros of f.

a. Suppose that one of these zeros, $\mathbf{x}_0^p$, is known. Then a new orbit can be predicted by minimizing the function:

$$\phi = g_1^2 + g_2^2 + g_3^2 + g_4^2 + \left((\delta x_{10})^2 + (\delta x_{30})^2 + (\delta\dot{x}_{20})^2 - \theta\right)^2,$$

where g_1, g_2, g_3 are the functions appearing in Eqs (3), θ denotes a proper small deviation along the family and

$$g_4 = \frac{\partial x_3}{\partial x_{10}}\delta x_{10} + \left(\frac{\partial x_3}{\partial x_{30}} - 1\right)\delta x_{30} + \frac{\partial x_3}{\partial \dot{x}_{20}}\delta\dot{x}_{20} + \frac{\partial x_3}{\partial t}\delta T.$$

The purpose of this construction of ϕ is: 1) to ensure that $g_1 = g_2 = g_3 = g_4 = 0$ are fulfilled and 2) to force the estimation of the new orbit to be at a distance θ from the known one.

b. If the approximation is not "good enough" it can be corrected by minimizing the function:

$$\psi = h_1^2 + h_2^2 + h_3^2,$$

where the functions h_1, h_2, h_3 are those appearing in System (4). This is, obviously, equivalent to the solution of this system.

The optimization can be made by any one of the usual methods (see, for example, the BFGS routine in [3] or the method of [7] which can be applied to problems with imprecise function values).

3. Numerical Applications and Concluding Remarks

The above mentioned family has been computed using both of the methods LT and OT. In Figure 1 we give the characteristic curve (x_{10}, x_{30}) of the family. Since the most time consuming part of the calculations is the integration of the equations of motion and the variations, we have estimated the power of each method according to the number of times this integration was required.

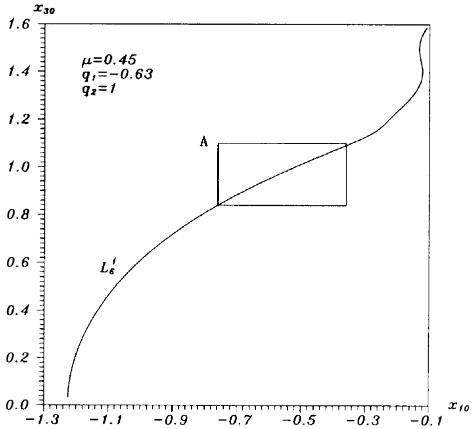

Figure 1. The characteristic curve (x_{10}, x_{30}) of the family L_6^1.

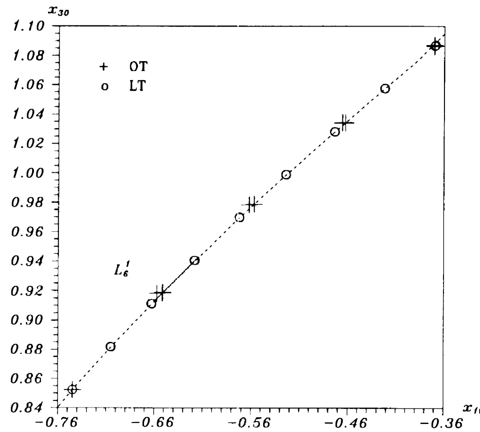

Figure 2. The behaviour of LT and OT in Box A of Figure 1. The symbols o and + denote the predictions and corrections of LT and OT respectively.

In Figure 2 we see a representation of the behaviour of the methods in a part of the family (part A of the characteristic curve given in Figure 1). The steps of the predictions of each method were increased to the maximum values so that the convergence of the correction iterations is preserved.

The total number of integrations performed in LT was 38 while that of OT was 24. We can see in Figure 2 that the advantage of the new method is that the rate of prediction steps is larger than this of LT.

Instead of using the LT or OT techniques for the correction step one may use also a method based on the notion of a "characteristic polyhedron" [1]. This method has the advantage that it does not depend on the variational equations whose integration sometimes inherents large errors, especially in the case of long period orbits or orbits which come close to singularities.

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