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A Convergence-Improving Iterative Method for Computing Periodic Orbits near Bifurcation Points

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The accurate computation of periodic orbits and the precise knowledge of their bifurcation properties are very important for studying the behavior of many dynamical systems of physical interest. In this paper, we present an iterative method for computing periodic orbits, which has the advantage of improving the convergence of previous Newton-like schemes, especially near bifurcation points. This method is illustrated here on a conservative, nonlinear Mathieu equation, for which a sequence of period-doubling bifurcations is followed, long enough to obtain accurate estimates of the two *universal scaling* constants α, β, as well as the *universal rate* δ, by which the bifurcation values of a parameter $q=q_k$, $k=1, 2, 3, ...,$ tend to their limiting value, $q_\infty < \infty$, as k increases. © 1990 Academic Press, Inc.

1. INTRODUCTION

In recent years, it has been widely recognized that even the simplest *nonlinear* dynamical systems of the form

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \qquad \mathbf{x} = (x_1, ..., x_n), \tag{1.1}$$

can have solutions (or orbits) $\mathbf{x}(t)$, with remarkable properties. Perhaps the most remarkable of them all is that, for large classes of initial conditions $\mathbf{x}(0)$, the orbits of (1.1) behave, as $t \to \infty$, in an apparently unpredictable, irregular or, as is more commonly called, *chaotic* way [1–5].

These chaotic orbits are located near *unstable* (hyperbolic) fixed points and periodic orbits, and are present at all scales, when (1.1) describes a non-integrable *Hamiltonian* system [5, 6]. On the other hand, again in the Hamiltonian case,

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around the *stable* (elliptic) periodic orbits there are "*islands*" of regular behavior whose size is *larger*, the *smaller* the period of the orbit [5–7].

In fact, periodic orbits are "dense" among all orbits of a Hamiltonian systems and—as Poincaré himself had suggested [8]—by studying them, one can understand some of the more "global" properties of the motion of dynamical systems [6–10].

In this paper, we present a rapidly convergent algorithm for calculating periodic orbits by computing iteratively, and to any desired accuracy, the coefficients A_n of their Fourier series expansions. This algorithm becomes especially significant near *bifurcation* points, where new periodic orbits appear, and other more traditional approaches (like Newton's method, etc.) cannot easily distinguish among the closely neighboring roots of the associated nonlinear *algebraic* equations for the A_n's.

Here we shall illustrate this method on the equation

$$\ddot{x} + (1 + 2q \cos 2t)\, x - x^3 = 0, \tag{1.2}$$

derived from the Hamiltonian

$$H = \tfrac{1}{2}(\dot{x}^2 + x^2) - \tfrac{1}{4}x^4 + qx^2 \cos 2t, \tag{1.3}$$

by studying its main sequence of *period-doubling* bifurcations of periodic orbits with period $T_k = 2^k\pi$, $k = 1, 2, \ldots$ as the parameter $q > 0$ in (1.2) is increased. The importance of such period doubling sequences as "routes" towards large scale chaotic behavior, in Hamiltonian as well as dissipative systems, has been amply discussed in the recent literature [2–5] and need not be repeated here.

It must be pointed out, however, that the method we shall describe in this paper represents a definite improvement over an iterative-variational method introduced by Eminhizer *et al.* [9] and Helleman and Bountis [6] to obtain periodic solutions of dynamical systems like (1.2). It is an improvement in the sense that it converges more rapidly than Helleman's Newton-like scheme [10], when applied to the period-doubling bifurcations of (1.2). The main reason behind this improvement is that our method of "root-searching" is not affected by a Jacobian taking small values in between neighboring roots, while Newton's method is notorious for breaking down precisely in that case.

Thus, we start in the next section by reviewing Budinsky's application of the more usual iterative Fourier schemes to the period-doubling bifurcations of Eq. (1.2) [7]. We also outline there briefly her stability analysis using Hill's determinants and discuss the accuracy limitations encountered already in determining the third bifurcation, i.e., that of the period 8 (in units of π) orbit out of the orbit of period 4.

Then, in Section 3, we describe our method for solving the nonlinear algebraic equations of Section 2 (for the Fourier coefficients A_n of these orbits) and present our results for the bifurcations of period 4, period 8, and period 16 orbits. Denoting by q_k the value of q at which the orbit of period 2^k *appears*, we compute, in Section 4, to high accuracy the ratios

$$\delta_k = (q_k - q_{k+1})/(q_{k+1} - q_{k+2}), \qquad k = 1, 2, 3, \ldots \tag{1.4}$$

and verify that they quickly tend to the universal number $\delta = 8.7210972\ldots$ [11, 12, 3, 4] as k increases.

Moreover, in Section 4, we compute the first approximations of the universal *scaling constant*,

$$\alpha = \lim_{n \to \infty} \frac{d_n}{d_{n+1}} = 4.0180767\ldots, \tag{1.5}$$

where d_n is the distance of the two nearest points of a periodic orbit of period 2^n (when it bifurcates to a period 2^{n+1} orbit), as well as approximations of the second universal scaling constant $\beta = 16.36389\ldots$ [12]. All these values turn out to agree very well with those given in the literature [11, 12] for area-preserving mappings in the plane.

We finally end, in Section 5, with some concluding remarks on the applicability of our methods to the bifurcation properties of periodic orbits of more general dynamical systems.

2. An Iterative Fourier Method for Period-Doubling Bifurcations

In this section, we describe an iterative Fourier scheme for obtaining periodic solutions of the nonlinear Mathieu equation

$$\ddot{x} + (1 + 2q \cos 2t)\, x - x^3 = 0, \tag{2.1}$$

of period $T_k = 2^k\pi$, $k = 1, 2, 3, \ldots$. These solutions (or, orbits) bifurcate out of one another at the values q_k, with $0 < q_1 < q_2 < \cdots < q_\infty < \infty$, at which the orbit of period T_{k-1} *destabilizes* and gives "birth" to a *stable* orbit of period T_k.

At $q = 0$, the origin of the phase plane x, $\dot{x}$ is a stable fixed point, surrounded by "elliptic" closed curves, as depicted schematically in Fig. 1a. At $0 < q \ll 1$, this point has become unstable and a stable period 2 (in units of π) orbit has appeared, intersecting at the points I_1, I_2 the *surface of section* [1–4],

$$\textstyle\sum_0 = \{(x(t_n), \dot{x}(t_n))/t_n = n\pi,\ n \in Z\}, \tag{2.2}$$

see Fig. 1b.

Following these points by solving (2.1) numerically (e.g., by a standard Runge–Kutta-type scheme) one finds that they turn unstable at

$$q = q_1 = 0.501041950415, \tag{2.3}$$

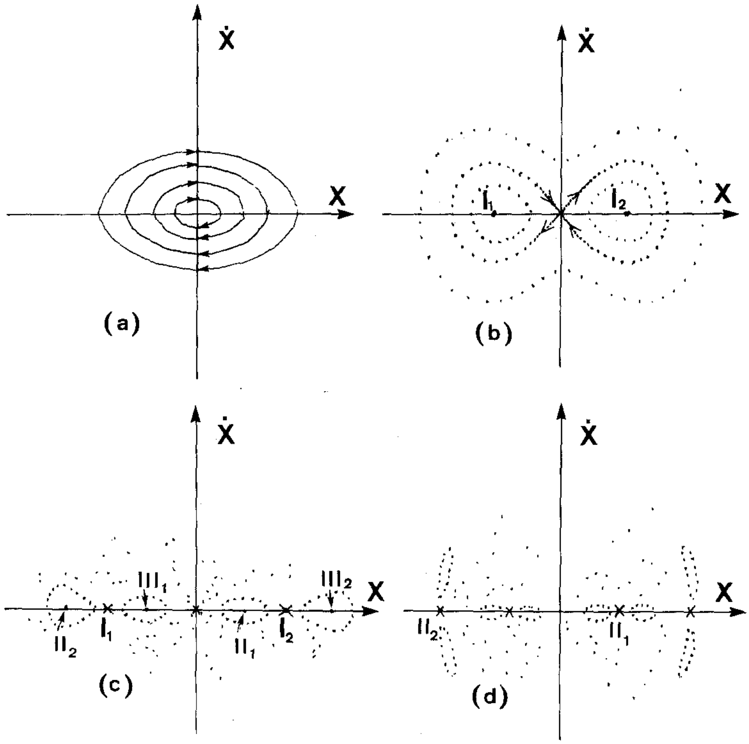

FIG. 1. A schematic drawing of the intersections of orbits of the nonlinear Mathieu equation (2.1) with the $x, \dot{x}$ surface of Section (2.2), at $t = n\pi$, $n = 0, 1, 2, 3...$: (a) $q = 0$, the integrable case, where all orbits lie on the plane; (b) $0 < q \ll 1$, where there is a stable period 2π orbit labeled by I; (c) $q = 0.51$, where orbit I has destabilized and given "birth" to the stable period 2π orbits II and III; (d) $q = 0.526$, where orbits II and III have also turned unstable, yielding two stable period 4π orbits.

giving rise to *a pair* of period 2 orbits, labeled by II and III in Fig. 1c, which are not *each* symmetric w.r.t. the $\dot{x}$ axis, but do possess this symmetry w.r.t. each other (Note that due to the form of Eq. (2.1), if $x(t)$, $\dot{x}(t)$ is a solution, then so is $-x(-t)$, $\dot{x}(-t)$.)

We may thus follow one of these period 2 orbits (e.g., the one labeled by II) and find that it also destabilizes at

$$q_1 < q_2 = 0.5250750359375, \tag{2.4}$$

(as does orbit III, of course) but in a rather peculiar way: Point II_1 splits into two "islands" along the x-axis, while II_2 into two much thinner islands nearly vertically "off" the horizontal axis, see Fig. 1d. This is precisely the period-doubling scenario observed in conservative models of two degrees of freedom [11, 12] occurring in much the same way as Feigenbaum discovered first for one-dimensional systems [13].

Of course, by the time period 4 orbits have appeared at $q = q_2$, large scale chaos has already spread in phase space, as shown schematically in Figs. 1c, d. Even though this is the typical situation, it is still interesting to develop methods to further pursue period-doubling bifurcations for several reasons: First, much less is known about their general properties in *higher-dimensional* systems [14, 15]. Second, there are similar bifurcation phenomena of orbits of much longer period which occur while there is still large scale regular motion in phase space. And finally, one may wish to accurately verify certain universality properties, which are expected to hold as $k \to \infty$, or $q \to q_\infty < \infty$.

In any case, to construct periodic orbits of Eq. (2.1) as convergent Fourier series of the form

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\nu_r t}, \qquad A_n^* = A_{-n}, \tag{2.5}$$

one starts by specifying the important "winding" or "rotation number" σ [6, 9, 10] defined by

$$\sigma = m_1/m_2 = \nu_1/\nu_2 \tag{2.6}$$

as the ratio of two fundamental frequencies of the problem

$$\nu_1 = m_1 \nu_r, \qquad \nu_2 = m_2 \nu_r, \tag{2.7}$$

where m_1, m_2 are positive integers and ν_r is the recurrence (or, actual) frequency of the orbit.

In the case of our period-doubling sequence of orbits with period

$$T_k = 2^k \pi, \qquad \nu_r = 2^{1-k}, \qquad k = 1, 2, 3, ..., \tag{2.8}$$

the second frequency of the problem $\nu_2 = 2$, i.e., is that of the periodic driving term in (2.1), and hence $m_2 = 2^k$ is known for each orbit, cf. (2.7), (2.8). Moreover, since the period 2^k orbit intersects the surface of Section (2.2) m_2 times, "rotating" around the origin m_1 times, its $m_1 = 2^{k-1}$ value is also known. Thus, expecting that the main Fourier coefficient in (2.5) will be A_{m_1}, one may write (2.1) as a forced harmonic oscillator of frequency ν_1,

$$\ddot{x} + \nu_1^2 x = \nu_1^2 x - (1 + 2q \cos 2t)\, x + x^3 \tag{2.9}$$

and substitute all of the above in (2.9) to obtain a recursion relation for the A_n's [7],

$$(m_1^2 - n^2)\, \nu_r^2 A_n' = (\nu_1^2 - 1)\, A_n + \sum_{n_1+n_2+n_3=n}\sum\sum A_{n_1} A_{n_2} A_{n_3} - q(A_{n+m_2} + A_{n-m_2}) \tag{2.10}$$

which are to be solved for the A_n's from the l.h.s. This can be done for all n except $n = m_1$, for which either the A_{m_1} is solved from the r.h.s. of (2.10) or, one scales the

A_n's by defining $B_n = A_n/x(0)$ and solves the $n = m_1$ equation for the initial condition $x(0)$, obtaining B_{m_1} from the $t = 0$ equation

$$1 = B_0 + \sum_{n=-\infty}^{\infty} B_n. \tag{2.11}$$

This program was carried out in [7] and the periodic orbits of period 2, 4, and 8 were convergently obtained, but not without making one further modification: A term of the form βA_n had to be added to both sides of (2.10), where β was a suitably chosen constant. Moreover, in the case of period 8, β had to be assigned different values β_1 and β_2 for the iteration of the even and odd n coefficients, respectively. Still, despite these modifications, the A_n's for period 8 could only be calculated with limited accuracy that prevented a satisfactory calculation, e.g., of the universal rates of this problem [7].

Now, as it has been argued elsewhere [10], the above iteration scheme can converge quadratically, i.e., is *Newton-like*, provided the initial values of the A_n's are "close enough" to the final ones. And what if there are several possibilities of "final" values nearby, as it does happen just beyond a bifurcation point? This is exactly where a Newton-like scheme becomes problematic and a new iterative method needs to be introduced—like the one described in the next section—which will circumvent the convergence problems mentioned above.

Before describing this new method, however, and its results, we end this section with a brief discussion of how the knowledge of the Fourier coefficients A_n can be used to study the stability properties of the associated periodic orbit. This is done using Floquet theory and Hill's determinants [16, 17] in the following way: Suppose we want to determine the stability of a periodic orbit $\hat{x}(t)$ of the form (2.5). We first linearize Eq. (2.1) about this orbit, substituting $x = \hat{x} + z(t)$ and dropping $O(z^2)$ terms to find

$$\ddot{z}(t) + [1 + 2q \cos 2t - 3\hat{x}^2]\, z(t) = 0. \tag{2.12}$$

This is a Hill's equation [17] of the type

$$\ddot{z}(t) + Q(t)\, z(t) = 0 \tag{2.13}$$

with $Q(t) = Q(t + 2\pi/\nu_r)$, also expressed as a Fourier series

$$Q(t) \equiv 1 + 2q \cos 2t - 3\hat{x}^2 = \sum_{n=-\infty}^{\infty} a_n e^{in\nu_r t}, \tag{2.14}$$

and a_n's given explicitly in terms of the known A_n's of (2.5) by

$$a_n = \delta_{n,0} + q(\delta_{n,m_2} + \delta_{n,-m_2}) - 3 \sum_{k=-\infty}^{\infty} A_k A_{n-k}. \tag{2.15}$$

Now, the general solution of (2.13) is given by a linear combination of its fundamental solutions $z_{\pm}(t) = \exp(\pm i\mu t)\, P_{\pm}(t)$, where $P_{\pm}(t) = P_{\pm}(t + 2\pi/v_r)$ and μ is the so-called Floquet exponent [16, 17]. Thus, the boundedness (or unboundedness) of $z(t)$, and hence the stability (or, instability) of the periodic orbit $\hat{x}(t)$, depends on whether the Floquet exponent μ is real (or, imaginary). For an orbit of frequency v_r, cf. (2.8), this is decided finally by the criterion [16]:

$$|S_k| \equiv |1 - 2\sin^2(2^{k-1}\pi\sqrt{a_0}) \det \mathbf{D}| \begin{cases} <1\text{: Stability} \\ >1\text{: Instability,} \end{cases} \tag{2.16}$$

$k = 1, 2, ...,$ where $\mathbf{D}$ is the Hill's matrix, with elements

$$D_{m,n} = a_{n-m}/(a_0 - n^2 v_r^2), \qquad n \neq m$$

and

$$D_{m,m} = 1, \qquad n, m = -\infty, ..., \infty. \tag{2.17}$$

Thus, the bifurcation values q_k, like q_1 and q_2 of (2.3) and (2.4), are computed as follows: At $q = q_1$, $S_1 = 1$, whereupon it starts to decrease reaching -1 at $q = q_2$. Then, we evaluate S_2 using the coefficients of the period 4 orbit and determine q_3 from:

$$1 \geqslant S_2 \geqslant -1: \qquad q_2 \leqslant q \leqslant q_3 = 0.527780774375. \tag{2.18}$$

However, due to limited accuracy in the calculation of the A_n's of the period 8 orbit, q_4 could only be computed to 4-digit accuracy by the methods of this section. We, therefore, turn now to the new method of this paper to overcome these convergence difficulties and compute to the desired precision the bifurcations at $q = q_4, q_5$, etc.

3. A New Iterative Scheme with Improved Convergence

In order to circumvent the convergence difficulties of the Newton-like schemes of the previous section, we shall introduce here a new iterative method for solving the nonlinear algebraic equations (2.10). This method, which we call *nonlinear successive overrelaxation bisection method* (NSORB), is an extension of the well-known generalized linear iterative methods [18, 19] and has the advantage of not being affected by variations in the magnitude of the Jacobian, which is primarily what plagues the convergence rate of Newton schemes near bifurcation points.

Below, we briefly describe the main steps of the NSORB method. More details, convergence proofs, and further applications will be published elsewhere [20]. Suppose we have to solve a system of nonlinear algebraic equations

$$f_i(\mathbf{x}) = 0, \qquad \mathbf{x} = (x_1, ..., x_N), \qquad i = 1, 2, ..., N \tag{3.1}$$

with f_i: $\mathbb{R}^N \to \mathbb{R}$ continuous. Starting with the initial choice,

$$\mathbf{x}^0 = (x_1^0, x_2^0, \dots, x_N^0), \tag{3.2}$$

we shall obtain estimates $\mathbf{x}^k$, $k = 1, 2, \dots$, of the solution $\mathbf{x}$ of Eq. (3.1) by solving, at the kth step, for the component $x \equiv x_i^{k+1}$, from the equation

$$g_i(x) = f_i(x_1^k, \dots, x_{i-1}^k, x, x_{i+1}^k, \dots, x_N^k) = 0. \tag{3.3}$$

We then set the following, using a relaxation parameter $\omega \in (0, 1]$,

$$x_i^{k+l} = x_i^k + \omega(x - x_i^k), \qquad i = 1, \dots, N, \qquad k = 0, 1, \dots, \tag{3.4}$$

and compute x, performing m-steps of the following modified one-dimensional bisection method [21–24],

$$x_{\lambda+1} = x_\lambda + \operatorname{sgn} g_i(x_i^0) \operatorname{sgn} g_i(x_\lambda)\, h_i/2^{\lambda+1}, \qquad \lambda = 0, 1, 2, \dots, m-1; \tag{3.5}$$

cf. (3.3), where sgn Θ is the well-known sign function,

$$\operatorname{sign} \Theta = \begin{cases} -1, & \text{if } \Theta < 0 \\ 0, & \text{if } \Theta = 0 \\ 1, & \text{if } \Theta > 0, \end{cases} \tag{3.6}$$

and h_i is such that

$$\operatorname{sgn} g_i(x_i^0) \cdot \operatorname{sgn} g_i(x_i^0 + h_i) = -1. \tag{3.7}$$

It is easy to check that the above iterative scheme converges to the value of x that satisfies (3.3). Moreover, it can be shown that the number of iterations m needed to obtain x, from (3.5), with accuracy ε, is given by [21–24]

$$m = \lceil \log_2(h_i \varepsilon^{-1}) \rceil, \tag{3.8}$$

where $\lceil a \rceil$ denotes the least positive integer that is not smaller than the real number a. Also, the convergence of the iterates of (3.3)–(3.4) is similar to the convergence of the class of generalized linear iterative methods [18, 19], since these belong to that class. Finally, instead of the Jacobi iterations (3.3)–(3.4) we can use the relative Gauss–Seidel iterations. Thus, rather than solving Eq. (3.3), we can solve the following equation in the same manner,

$$g_i(x) = f_i(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x, x_{i+1}^k, \dots, x_N^k) = 0.$$

The m-step NSORB method described above is particularly suited for problems (3.1) whose zeroes are very close to each other, since, in that case, it is well known that Newton-like schemes do not always converge to the desired solution, even for initial points starting close to it. Thus, the NSORB method turns out to be especially useful near bifurcation points, where different periodic orbits coexist, within very small distances from each other in phase space, and correspond to different solutions $\{A_n, n=0, 1, 2, ...\}$ of the system

$$(1-n^2\nu_r^2)\,A_n+q(A_{n+m_2}+A_{n-m_2})-\sum_{n_1+n_2+n_3=n}\sum\sum A_{n_1}A_{n_2}A_{n_3}=0, \qquad (3.9)$$

cf. (2.10). This is particularly true if the bifurcated "daughter" periodic orbit has the same or double the period of the "mother" orbit, since then both orbits have A_n's that satisfy equations (3.9), with the same ν_r. In period-doubling, for example, the "mother" orbit can always be recovered, at the ν_r of its "daughter," by setting all the odd n coefficients A_n of the "daughter" orbit equal to zero.

Furthermore note that, with the proper choice of h_i in (3.5), it is possible to isolate a solution of (3.3) (or (3.9)) without the necessity of a good initial estimate (3.2). Finally, since in the various function evaluations, only the signs of these functions need to be correct, the NSORB method can be applied to problems where the actual values of these functions are not known to great precision.

We now proceed to discuss the results of the application of the NSORB method to the period-doubling bifurcations of Eq. (2.1) discussed in the previous section: As a starting point, we began by evaluating, at different values of q, the Fourier coefficients of the periodic orbit II of period 2, as well as the coefficients of the period 4 and period 8 orbits that bifurcate out of it (see Figs. 1c, d).

The first 16 of these coefficients are listed below in Table I. They correspond, respectively, to q values, at which these orbits are ready to bifurcate to their "daughter" periodic orbits of twice the period and have been obtained (to an accuracy of 10^{-15}) by iterating Eqs. (3.3)–(3.5) some 50–60 times on a personal computer.

Observe that, as expected from period-doubling bifurcations of other dynamical systems [1–5], these orbits intersect the surface of Section (2.2) at closely neighboring points. This can be verified here by using the obtained A_n's to substitute in the Fourier series (2.5) at $t=k\pi$, with $\nu_r=1$, 1/2, and 1/4 for the three periodic orbits of Table I. Note the close proximity of their A_0 values, and the values of the important coefficients A_1, A_2, and A_4 of the period 2π, 4π, and 8π orbits, respectively.

Observe, more generally, how the A_n values of these orbits compare, for n even: As predicted by the theory [25], the A_n values of one orbit are very close to the A_{2n} values of the orbit that has bifurcated out of it. Thus, even though the magnitude of the A_n's generally decreases rapidly with increasing n, as the period of the orbit gets higher, large A_n's will appear at higher and higher n (see, e.g., the A_{12} coefficient of period 8π). This implies that an accurate computation of higher period orbits, by this method, requires the knowledge of an exponentially growing number of Fourier coefficients A_n. Since, according to the above remarks, the magnitude of these coefficients does not significantly change with increasing period, one can estimate the truncation index $n_{\max}$ necessary to achieve a desired accuracy.

TABLE I

	Period 2π (II) orbit	Period 4π orbit	Period 8π orbit
q	0.5250750359375	0.527780774375	0.528089535489
A_0	−0.101079850572	−0.102084233567	−0.102478551216
A_1	0.405880573247	0.112950375934 E-1	−0.205457274490 E-2
A_2	0.215024973106 E-3	0.406432413099	0.114319030915 E-1
A_3	0.162272390174 E-1	0.120475694919 E-2	−0.752684006155 E-3
A_4	0.249351465912 E-3	0.144233907811 E-3	0.406442830989
A_5	0.974953318166 E-5	0.416106833246 E-4	−0.272002262271 E-3
A_6	0.194410572006 E-5	0.163382954916 E-1	0.122135978580 E-2
A_7	−0.648782646554 E-5	−0.286450436909 E-4	−0.351572017672 E-4
A_8	−0.180318836485 E-6	0.253321521252 E-3	0.136194204310 E-3
A_9	−0.581424467648 E-7	−0.223803586696 E-4	−0.546269657348 E-5
A_{10}	−0.348385768741 E-8	0.106143802603 E-4	0.425123798715 E-4
A_{11}	0.199619373644 E-8	−0.228511870199 E-5	0.906310344601 E-6
A_{12}	0.649999927290 E-10	0.210237043064 E-5	0.163513053801 E-1
A_{13}	0.445903971647 E-10	−0.297437291704 E-6	0.182232225575 E-5
A_{14}	0.277474320169 E-11	−0.657893753635 E-5	−0.289728555589 E-4
A_{15}	−0.299587599489 E-12	−0.586867940003 E-8	0.563888239769 E-5
A_{16}	−0.781397215670 E-15	−0.182883419201 E-6	0.254487597637 E-3

In particular, for period 8π orbits, we have used $n_{\max} = 36$ and for period 16π orbits $n_{\max} = 56$, which guarantee an accuracy better that 10^{-8} and 10^{-6}, respectively, for the location of these orbits in phase space, at all t.

4. The Computation of Universal Constants

As we saw in the previous sections, the accuracy of the computation of periodic orbits near their bifurcation points is indeed a delicate matter. In a period-doubling sequence, like the one we have followed in this paper, several solutions of Eq. (3.9) can exist, very close to each other, for the same v_r values.

Moreover, at higher and higher periods, more and more coefficients A_n (and consequently larger and larger determinants in (2.16)) must be calculated to permit a highly accurate computation of the periodic orbits and their bifurcation values. Thus, after obtaining the period 8π orbits on a modest UNIVAC 1100/60, we use a supercomputer IBM-3090 600E for our remaining calculations of the orbits of period 16π.

These accurate results were first used to determine to 12-digit precision the bifurcation values q_4 and q_5 at which the orbts of period 16π and 32π, respectively, first appear:

$$q_4 = 0.528089535489, \qquad q_5 = 0.528124936353. \tag{4.1}$$

Denoting then by δ the rate at which these values tend to their limit q_∞, i.e.,

$$q_k - q_\infty \propto \delta^{-k}, \qquad k \ \text{large}, \tag{4.2}$$

we compute the ratios

$$\delta_k = (q_k - q_{k+1})/(q_{k+1} - q_{k+2}), \tag{4.3}$$

for $k = 1, 2, 3$, using the numbers listed in (2.3), (2.4), (2.8), and (4.1) above, and find

$$\delta_1 = 8.88226488910$$

$$\delta_2 = 8.76320985647$$

$$\delta_3 = 8.72185249490.$$

This sequence indeed appears to indicate that, as k increases, the δ_k's quickly tend to the universal value $\delta = 8.72109720...$, as expected from other studies on similar conservative systems [11, 12].

There are two more universal constants associated with period-doubling bifurcations of our Eq. (2.1). They correspond to scaling properties of these orbits and are computed as follows:

Denote by d_k the distance between the two points of the period 2^k orbit, which lie on the x-axis of the Poincaré surface of section (see Table II), at the $q = q_k$ value of its bifurcation to an orbit of twice the period. It is expected, from other similar studies that

$$\alpha_k = d_k/d_{k+1} \xrightarrow[k \to \infty]{} \alpha = 4.0180767..., \tag{4.4}$$

where this value value of α is *universal* for conservative two-degree of freedom Hamiltonian systems. From our results on the nonlinear Mathieu equation (2.1) we compute

$$\alpha_1 = d_1/d_2 = 4.0219742866$$

$$\alpha_2 = d_2/d_3 = 4.0182210698$$

which indeed appear to tend rather quickly to the universal value (4.4).

The universality of the above two constants α, δ (with different values than were found above) was first observed for *dissipative* systems, whose dynamics is one-dimensional [13]. In the case of conservative systems, on the other hand (like area-preserving mappings in the plane), a third universal constant was found, corresponding to *scaling* of the orbits in a direction *vertical* to their axis of symmetry (for our nonlinear Mathieu equation, this is the x-axis of the surface of section).

This third constant, β, was incorporated by MacKay—together with the α of

TABLE II*

Surface of Section Intersections of Periodic Orbits

	PII at $q=q_2$	P4 at $q=q_3$	P8 at $q=q_4$	P16 at $q=q_5$
$x(0)$	0.74407446	0.76923970	0.76297668	0.76454808
$\dot{x}(0)$	0.0	0.0	0.0	0.0
$x(\pi)$	−0.94436961	−0.94683466	−0.94879603	−0.94858823
$\dot{x}(\pi)$	0.0	−0.00790948	−0.00702482	−0.00735205
$x(2\pi)$	0.74407446	0.71928847	0.71862038	0.71813724
$\dot{x}(2\pi)$	0.0	0.0	0.00048640	0.00043174
$x(3\pi)$	−0.94436961	−0.94683466	−0.94578401	−0.94591482
$\dot{x}(3\pi)$	0.0	0.00790948	0.00897732	0.00908537
$x(4\pi)$	0.74407446	0.76923970	0.77539626	0.77557648
$\dot{x}(4\pi)$	0.0	0.0	0.0	0.00002964
$x(5\pi)$	The above points are cyclically repeated	The above points are cyclically repeated	−0.94578401	−0.94572314
$\dot{x}(5\pi)$			−0.00897732	−0.00896579
$x(6\pi)$			0.71862038	0.71888830
$\dot{x}(6\pi)$			−0.00048640	−0.00055252
$x(7\pi)$			−0.94879603	−0.94914647
$\dot{x}(7\pi)$			0.00702482	0.00674556
$x(8\pi)$			0.76297668	0.76145726
$\dot{x}(8\pi)$			0.0	0.0

* The numbers listed in this section (and on Table I) have been rounded off. They are known to several more digits than is shown here.

(4.4)—in a renormalization analysis of period-doubling in such two-dimensional systems [12] and was found to have the value

$$\beta = 16.363896879.... \tag{4.5}$$

Our equation (2.1) also belongs to this class, since its dynamics can be represented by a locally area-preserving Poincaré map on its surface of Section (2.2) [4].

We have thus denoted by b_k, at the $q = q_k$ value where the period 2^{k+1} orbit is born, the distance between the two points of the period 2^k orbit, that had split off the x-axis at the previous bifurcation, and computed, using Table II,

$$\beta_1 = b_1/b_2 = 16.2611705$$

$$\beta_2 = b_2/b_3 = 16.4124209.$$

These values again appear to approach the universal rate (4.5).

5. Concluding Remarks

We have described a numerical method for accurately computing the Fourier coefficients of periodic orbits of dynamical systems, which we have called the NSORB method (nonlinear successive overrelaxation bisection). This method complements the usual variational-iterative schemes for such orbits in that it improves their convergence near bifurcation points.

We have applied NSORB here to a period-doubling sequence of periodic solutions of a conservative nonlinear Mathieu equation (2.1) and have succeeded in overcoming the difficulties of previous schemes, and determining (to any desired accuracy) the first few bifurcations of period 2^k orbits $k = 1, 2, 3, \ldots$.

As a result, we have been able to compute the first few approximations of the universal constants α, β, and δ and found that they quickly tend to their expected values for this class of problems. These values have been long known from work on are-preserving mappings, but have not been as often (and as accurately) computed for conservative differential equations, due to serious difficulties of numerical precision.

Of course, we could apply the NSORB scheme to higher order period doubling bifurcations and calculate even better approximations to the universal constants of this paper, obtaining more digits of the numbers known already from area-preserving maps. Instead, we prefer to turn our attention, in future publications, to other problems of bifurcations of periodic orbits in dynamical systems.

In more than two degrees of freedom Hamiltonian systems, for example, the results are a lot more sporadic and their generality far are from being established. Infinite period-doubling sequences have apparently been observed only within one somewhat restricted class of conservative 4-dimensional mappings [15]. Using NSORB we could follow period-doubling sequences in simple three-degrees of freedom models and see whether they always terminate after a small number of bifurcations as some other researchers have suggested [14].

Finally, since by the NSORB approach, problems of small divisors [4] do not arise, it might be possible to use it to calculate the Fourier coefficients $A_{m,n}$ of *quasiperiodic* orbits, for which the rotation number σ, cf. (2.6), is *irrational*. Then, from the behavior of these $A_{m,n}$, as a function of some parameter of the equations, one could study the convergence properties of the series expansion of the solution and from that the "break up" of quasiperiodic orbits, when these series begin to diverge [26].

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