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AN ERROR ESTIMATION FOR THE METHOD OF BISECTION IN $\mathbb{R}^n$

BY

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ABSTRACT. *A formula for the Euclidean distance of each vertex of an m-simplex in $\mathbb{R}^n$ from its barycenter is given. This is used to derive an upper bound for the Euclidean distance of any point in an m-simplex in $\mathbb{R}^n$ from its barycenter. Finally we apply the above result to obtain an error estimate for the method of Bisection in $\mathbb{R}^n$ (applied to root-finding) and we give a proof of convergence of this method.*

1. Introduction.

Let $S_0^m = \langle X_0, X_1, \dots, X_m \rangle$ be an m-simplex (generalized triangle) in $\mathbb{R}^n$. We "bisect" simplexes as follows. We choose a longest edge, say $\langle X_i, X_j \rangle$, of S_0^m; calculate its midpoint $Y = (X_i + X_j)/2$ and define two new m-simplexes S_{11}^m and S_{12}^m, by replacing X_i by Y and X_j by Y respectively. Next, we bisect each S_{1k}^m, $k = 1, 2$, to form four new m-simplexes S_{2k}^m, $k = 1, 2, 3, 4$, and so on. This "generalized bisection" process has a number of applications; see, e.g. [4], [5], [7], [10], [11], [15]. In addition, several methods for the solution of a system of nonlinear equations

(1.1) $F^n(X) = \Theta^n$, where $F^n = (f_1, f_2, \dots, f_n)^T : S^n \to \mathbb{R}^n$ is continuous

$$\text{and} \quad \Theta^n = (0, 0, \dots, 0)^T \in \mathbb{R}^n$$

which implement the "generalized bisection" process have been proposed in the past few years, [5], [7], [11].

Define now the diameter D_p to be the length of the longest edge among the edges of an m-simplex S_p^m which is formed after p repeated iterations of the "generalized bisection" process applied to an original m-simplex S_0^m in $\mathbb{R}^n$. In [6, Theorem 3.1] an explicit bound is obtained for the rate of convergence of D_p:

$D_p \leqslant (\sqrt{3}/2)^{\lfloor p/m \rfloor} D_0$, where $\lfloor p/m \rfloor$ is the largest integer less than or equal to p/m. Better estimates of the rate of convergence of D_p and criteria of convergence of the "generalized bisection" process have been given in [1], [9], [12], [14] for triangles.

In this paper, we first prove that the Euclidean distance of each vertex X_i, $0 \leqslant i \leqslant m$ of S^m from its barycenter K^m is given by

$$\|X_i - K^m\|_2 = \frac{m}{m+1} \left\{ \frac{1}{m} \sum_{\substack{\ell=0 \\ \ell \neq i}}^{m} \|X_i - X_\ell\|_2^2 - \frac{1}{m^2} \sum_{\substack{k=0 \\ k \neq i}}^{m-1} \sum_{\substack{\ell=k+1 \\ \ell \neq i}}^{m} \|X_k - X_\ell\|_2^2 \right\}^{1/2}$$

This is used to derive un upper bound for the Euclidean distance of any point in S^m from K^m. Then we apply the above results to obtain an error estimate for the "generalized bisection" method applied to root-finding. To do this we consider the barycenter of S^n as an approximate solution of the system (1.1), and next, we prove that after p repeated iterations of the "generalized bisection" method, applied to an original n-simplex S_0^n, the error for the approximate solution obtained as described above does not exceed $n\{D_p^2 - (n-1)\, M_p^2 / 2n\}^{1/2} / (n+1)$, where M_p is the length of the smallest edge of S_p^n. It is also shown that this error estimate is not greater than $n(\sqrt{3}/2)^{\lfloor p/n \rfloor} D_0 / (n+1)$. Moreover a proof of convergence of the above method is given.

2. Preliminaries.

Definition 2.1. *The points* $X_0, X_1, \ldots, X_m$ *in* $\mathbb{R}^n$ *are said to be linearly independent if the vectors* $X_i - X_0$, $i = 1, 2, \ldots, m$ *are linearly independent. An m-simplex in* $\mathbb{R}^n$, $0 \leqslant m \leqslant n$, *is the closed convex hull of* $m+1$ *linearly independent points in* $\mathbb{R}^n$ *called its vertices ([2], [3], [6], [7], [11], etc.).*

We shall denote m-simplexes by $S^m = <X_0, X_1, \ldots, X_m>$ where the set $\{X_0, X_1, \ldots, X_m\}$ determines the set of its vertices.

Definition 2.2. *If* $S^m = < X_0, X_1, \ldots, X_m >$ *is an m-simplex in* $\mathbb{R}^n$ *then the (m−1)-simplex* $< X_0, X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m >$ *will be called the i-th face of* S^m, *while the 1-simplex* $< X_k, X_j >$, $0 \leqslant k < j \leqslant m$ *will be called edge of* S^m, *([3], [5], [6], [7], etc.).*

Definition 2.3. *The diameter D, of an m-simplex S^m in $\mathbb{R}^n$ is the length of the largest edge of S^m, while the microdiameter M of S^m is the length of the smallest edge of S^m, where the Euclidean norm is used to measure distances.*

We shall denote the Euclidean distance between the points A and B by d(A, B).

Definition 2.4. *([5], [6], [7], [11]). Let $S_0^m = < X_0, X_1, ..., X_m >$ be an m-simplex in $\mathbb{R}^n$, let $< X_i, X_j >$ be the/a largest edge of S_0^m and let $Y = (X_i + X_j) / 2$ be the midpoint of $< X_i, X_j >$. Then the bisection of S_0^m is the ordered pair of the m-simplexes (S_{11}^m, S_{12}^m) where*

$$S_{11}^m = < X_0, X_1, ..., X_{i-1}, Y, X_{i+1}, ..., X_j, ..., X_m >$$
$$\text{and} \quad S_{12}^m = < X_0, X_1, ..., X_i, ..., X_{j-1}, Y, X_{j+1}, ..., X_m >$$

The m-simplexes S_{11}^m and S_{12}^m will be called lower simplex and upper simplex respectively corresponding to S_0^m, while both S_{11}^m and S_{12}^m will be called elements of the bisection of S_0^m.

Definition 2.5. *Suppose that S_0^n is an n-simplex which includes at least one solution of the system of equations (1.1). Suppose further that (S_{11}^n, S_{12}^n) is the bisection of S_0^n and that there is at least one solution of the system (1.1) in some of its elements. Then this element will be called selected n-simplex produced after one bisection of S_0^m and it will be denoted by S_1^n. Moreover, if there is at least one solution of the system (1.1) in both elements, then the selected n-simplex will be the lower simplex corresponding to S_0^n. Suppose now that the bisection is applied with S_1^n replacing S_0^n giving thus the S_2^n. Suppose further that this process continues for p iterations. Then we call S_p^n the selected n-simplex produced after p iterations of S_0^n.*

Notation 2.6. The existence, of at least one solution of the system of equations (1.1) in the interior of an oriented n-simplex S_0^n, ([2], [3], [6], [7], [11], etc.), can be secured using the non-zero value of the topological degree of F^n at Θ^n relative to S_0^n, ([2], [3], [4], [5], [8], [11], etc.), denoted deg (F^n, S_0^n, Θ^n). More specifically if $F^n : S_0^n \rightarrow \mathbb{R}^n$ is continuous and $F^n(X) \neq \Theta^n$ anywhere on the oriented boundary $b(S_0^n)$, ([2], [3], etc.), of

S_0^n, then deg (F^n, S_0^n, Θ^n) is defined. Consequently, if deg $(F^n, S_0^n, \Theta^n) \neq 0$ then by Kronecker's theorem, [8], the system of equations (1. 1) has at least one solution in the interior of S_0^n. Suppose now that (S_{11}^n, S_{12}^n) is the bisection of S_0^n and that there are not solutions of the system (1.1) on $b(S_{11}^n)$ or $b(S_{12}^n)$. Then the topological degree is additive, i.e.

$$\deg (F^n, S_0^n, \Theta^n) = \deg (F^n, S_{11}^n, \Theta^n) + \deg (F^n, S_{12}^n, \Theta^n).$$

Consequently, deg $(F^n, S_1^n, \Theta^n) \neq 0$ and the process continues with S_1^n replacing S_0^n, and so on. Note that the topological degree can be easily evaluated by means of the algorithms described in [5], [7], [11] and [13].

3. Results.

Definition 3.1. *Let $S^m = < X_I, X_1 ,..., X_m >$ be an m-simplex in $\mathbb{R}^n$, $0 \leqslant m \leqslant n$. Then the barycenter of S^m, ([2], [3], etc.), denoted K^m, is a point in $\mathbb{R}^n$ such that*

$$K^m = \frac{1}{m+1} \sum_{i=0}^{m} X_i.$$

Remark 3.2. By convexity, the barycenter of any m-simplex S^m in $\mathbb{R}^n$ is an interior point of S^m.

Definition 3.3. *Let $S^m = < X_0, X_1 ,..., X_m >$ be an m-simplex in $\mathbb{R}^n$, $0 \leqslant m \leqslant n$, and let K^m be its barycenter. Then the radius of S^m, denoted A^m, is an 1-simplex in $\mathbb{R}^n$, whose vertices are the K^m and a vertex X_a of S^m, such that $d(X_a, K^m) = \max_{0 \leqslant i \leqslant m} \{(d\ (X_i, K^m))\}$. The distance $d(X_a, K^m)$ is called length of the radius of S^m.*

Notation 3.4. Let $S^m = < X_0, X_1 ,..., X_m >$ be an m-simplex in $\mathbb{R}^n$, $0 \leqslant m \leqslant n$, and let $T_i^{m-1} = < X_0, X_1 ,..., X_{i-1}, X_{i+1} ,..., X_m >$, $0 \leqslant i \leqslant m$, be the i-th face of S^m. Then the barycenter of T_i^{m-1} will be denoted by K_i^{m-1} and will be given by $\frac{1}{m} \sum_{\substack{k=0 \\ k \neq i}}^{m} X_k$. Moreover, if $T_{ij}^{m-2} = < X_0, X_1 ,..., X_{i-1}, X_{i+1} ,..., X_{j-1}, X_{j+1} ,..., X_m >$, where $0 \leqslant i < j \leqslant m$ is the j-th face of T_i^{m-1} then its barycenter will be denoted by K_{ij}^{m-2} and will be given by $\frac{1}{m-1} \sum_{\substack{k=0 \\ k\neq i,j}}^{m} X_k$.

Lemma 3.5. *Let* $S^m = < X_0, X_1, \dots, X_m >$ *be an m-simplex in* $\mathbb{R}^n$, $0 \leqslant m \leqslant n$, *and let* K^m *be its barycenter. Suppose that* X_i *is any vertex of* S^m *and that* K_i^{m-1} *is as in Notation 3.4. Then the following are valid*

(a) The points X_i, K^m *and* K_i^{m-1} *are collinear points.*

(b) $d(X_i, K^m) = \frac{m}{m+1} d(X_i, K_i^{m-1})$.

Proof. To prove (a), we consider the vectors $V_1 = K^m - X_i$ and $V_2 = K_i^{m-1} - X_i$. Also, we set $X_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})^T$ where $x_{k,p} \in \mathbb{R}$ for $0 \leqslant k \leqslant m$ and $1 \leqslant p \leqslant n$. Then

$$V_1 = \left\{ \left[\frac{1}{m+1} (x_{0,1} + x_{1,1} + \dots + x_{m,1}) - x_{i,1} \right], \dots, \left[\frac{1}{m+1} (x_{0,n} + x_{1,n} + \dots + x_{m,n}) - x_{i,n} \right] \right\}^T$$

or

$$(3.1)\ V_1 = \left\{ \frac{1}{m+1} \left[x_{0,1} + x_{1,1} + \dots + x_{i-1,1} - mx_{i,1} + x_{i+1,1} + \dots + x_{m,1} \right], \dots, \frac{1}{m+1} \left[x_{0,n} + x_{1,n} + \dots + x_{i-1,1} - mx_{i,n} + x_{i+1,n} + \dots + x_{m,n} \right] \right\}^T$$

and

$$V_2 = \left\{ \left[\frac{1}{m} (x_{0,1} + x_{1,1} + \dots + x_{i-1,1} + x_{i+1,1} + \dots + x_{m,1}) - x_{i,1} \right], \dots, \left[\frac{1}{m} (x_{0,n} + x_{1,n} + \dots + x_{i-1,n} + x_{i+1,n} + \dots + x_{m,n}) - x_{i,n} \right] \right\}^T$$

or

$$(3.2)\ V_2 = \left\{ \frac{1}{m} \left[x_{0,1} + x_{1,1} + \dots + x_{i-1,1} - mx_{i,1} + x_{i+1,1} + \dots + x_{m,1} \right], \dots,$$

$$\frac{1}{m}\left[x_{0,n} + x_{1,n} + \dots + x_{i-1,n} - mx_{i,n} + x_{i+1,n} + \dots + x_{m,n} \right] \Big\}^T$$

Combining (3. 1) and (3. 2) gives

$$(m + 1)\, V_1 - m\, V_2 = \Theta^n,$$

which proves the part (a) of the lemma.

To prove (b) we implement the following relationships

$$\left[d(X_i, K^m) \right]^2 = \sum_{p=1}^{n} \left[\frac{1}{m+1} (x_{0,p} + \dots + x_{i-1,p} - mx_{i,p} + x_{i+1,p} + \dots + x_{m,p}) \right]^2$$

or

$$(3.\ 3) \quad \left[d(X_i, K^m) \right]^2 = \frac{1}{(m+1)^2} \sum_{p=1}^{n} (x_{0,p} + \dots + x_{i-1,p} - mx_{i,p} + x_{i+1,p} + \dots + x_{m,p})^2$$

and

$$\left[d(X_i, K_i^{m-1}) \right]^2 = \sum_{p=1}^{n} \left[\frac{1}{m} (x_{0,p} + \dots + x_{i-1,p} - mx_{i,p} + x_{i+1,p} + \dots + x_{m,p}) \right]^2$$

or

$$(3.\ 4) \quad \left[d(X_i, K_i^{m-1}) \right]^2 = \frac{1}{m^2} \sum_{p=1}^{n} (x_{0,p} + \dots + x_{i-1,p} - mx_{i,p} + x_{i+1,p} + \dots + x_{m,p})^2$$

Combining (3. 3) and (3. 4) gives

$$(m + 1)\, d(X_i, K^m) = m\, d(X_i, K_i^{m-1})$$

which proves the part (b) of the lemma. □

Lemma 3.6. *Let $S^m = < X_0, X_1, \dots, X_m >$ be an m-simplex in $\mathbb{R}^n$, $0 \leqslant m \leqslant n$. Suppose that X_i and X_j, $i \neq j$ are any vertices of S^m and let K_i^{m-1} and K_{ij}^{m-2} be as in Notation 3. 4. Then the following are valid*

(a) The points X_j, K_i^{m-1} and K_{ij}^{m-2} are collinear points.

(b) $d\,(X_j, K_i^{m-1}) = \dfrac{m-1}{m}\; d\,(X_j, K_{ij}^{m-2})$.

Proof. The proof is similar to the proof of Lemma 3. 5. □

Lemma 3.7. *Let* $S^2 = < X_0, X_1, X_2 >$ *be a* $2-$ *simplex in* $\mathbb{R}^n$. *Let* X_k *be a point on any edge of* S^2, *say on* $< X_1, X_2 >$, *such that*

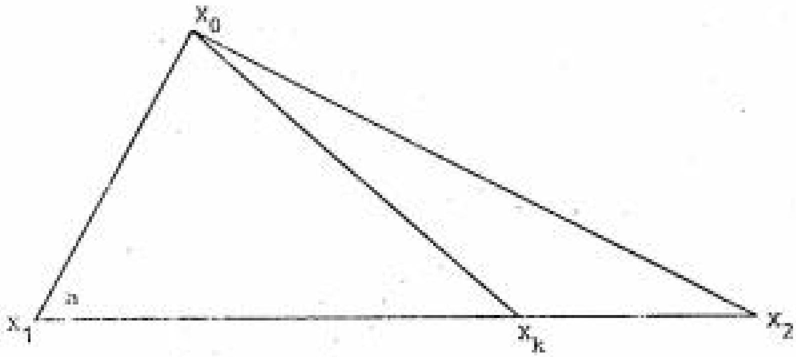
$$d(X_1, X_k) = \frac{\lambda - 1}{\lambda}\, d\,(X_1, X_2),$$

where $\lambda \in \mathbb{R}$ *and* $\lambda > 1$. *Then*

$$[d(X_0, X_k)]^2 = \frac{1}{\lambda}\,[d(X_0, X_1)]^2 + \frac{\lambda - 1}{\lambda}\,[d(X_0, X_2)]^2 - \frac{\lambda - 1}{\lambda^2}\,[d(X_1, X_2)]^2$$

Proof. We consider the interior angle a of the triangle $\Delta(X_0\, X_1\, X_2)$, as shown in Fig. 1.

Figure 1.



The law of consines in $\Delta(X_0\, X_1\, X_k)$ and $\Delta(X_0\, X_1\, X_2)$ yields

(3.5) $[d(X_0, X_k)]^2 = [d(X_0, X_1)]^2 + [d(X_1, X_k)]^2 - 2d(X_0, X_1)\, d(X_1, X_k)\ \cos a$

and

(3.6) $[d(X_0, X_2)]^2 = [d(X_0, X_1)]^2 + [d(X_1, X_2)]^2 - 2d(X_0, X_1)\, d(X_1, X_2)\ \cos a$

Combining (3.5) and (3.6) we get

$$[d(X_0, X_k)]^2 = [d(X_0, X_1)]^2 + [d(X_1, X_k)]^2$$

$$- \frac{d(X_1, X_k)}{d(X_1, X_2)}\,\{[d(X_0, X_1)]^2 + [d(X_1, X_2)]^2 - [d(X_0, X_2)]^2\}$$

or

$$[d(X_0, X_k)]^2 = \frac{1}{\lambda}\,[d(X_0, X_1)]^2 + [d(X_1, X_k)]^2$$

$$- \frac{\lambda - 1}{\lambda} [d(X_1, X_2)]^2 + \frac{\lambda - 1}{\lambda} [d(X_0, X_2)]^2$$

or

$$[d(X_0, X_k)]^2 = \frac{1}{\lambda} [d(X_0, X_1)]^2 + \frac{\lambda - 1}{\lambda} [d(X_0, X_2)]^2 - \frac{\lambda - 1}{\lambda^2} [d(X_1, X_2)]^2.$$

Thus the lemma is proven. □

Theorem 3.8. *Let* $S^m = < X_0, X_1, \ldots, X_m >$ *be an m-simplex in* $\mathbb{R}^n$, $0 \leqslant m \leqslant n$ *and let* K^m *be its barycenter. Then for each vertex* X_i, $0 \leqslant i \leqslant m$, *of* S^m *we obtain*

$$(3.7)\quad d(X_i, K^m) = \frac{m}{m+1} \left\{ \frac{1}{m} \sum_{\substack{\ell=0 \\ \ell \neq i}}^{m} [d(X_i, X_\ell)]^2 - \frac{1}{m^2} \sum_{\substack{k=0 \\ k \neq i}}^{m-1} \sum_{\substack{\ell=\kappa+1 \\ \ell \neq i}}^{m} [d(X_k, X_\ell)]^2 \right\}^{1/2}$$

Proof. The proof will proceed by induction on m. First assume that $S^1 = < X_0, X_1 >$ is an 1-simplex in $\mathbb{R}^n$, then for an arbitrary vertex of S^1, say X_0, we have

$$d(X_0, K^1) = \frac{1}{2} d(X_0, X_1),$$

which verifies (3.7).

Assume now that the theorem is true with $(m - 1)$ replacing m, where $m > 1$. So, assume that for an $(m - 1)$ — simplex S^{m-1} in $\mathbb{R}^n$ we have

$$(3.8)\quad d(X_i, K^{m-1}) = \frac{m - 1}{m} \left\{ \frac{1}{m-1} \sum_{\substack{\ell=0 \\ \ell \neq i}}^{m-1} [d(X_i, X_\ell)]^2 - \frac{1}{(m - 1)^2} \sum_{\substack{k=0 \\ k \neq i}}^{m-2} \sum_{\substack{\ell=\kappa+1 \\ \ell \neq i}}^{m-1} [d(X_k, X_\ell)]^2 \right\}^{1/2}, \quad \text{for} \quad 0 \leqslant i \leqslant (m - 1).$$

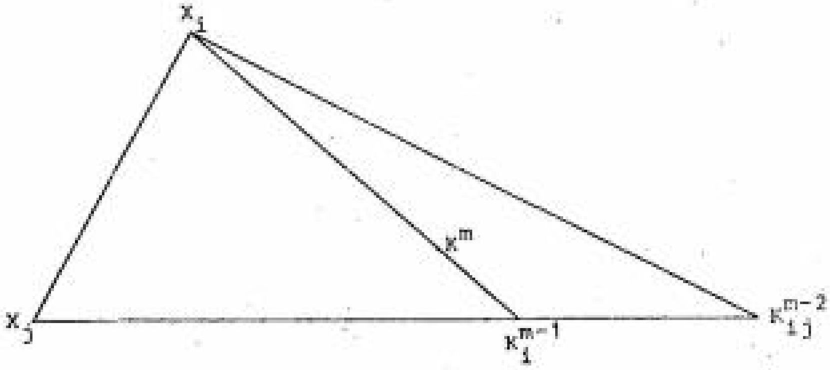
Also, from (3.8) using Lemma 3.5, we get

$$(3.9)\quad d(X_i, K_i^{m-2}) = \left\{ \frac{1}{m-1} \sum_{\substack{\ell=0 \\ \ell \neq i}}^{m-1} [d(X_i, X_\ell)]^2 - \frac{1}{(m - 1)^2} \sum_{\substack{k=0 \\ k \neq i}}^{m-2} \sum_{\substack{\ell=\kappa+1 \\ \ell \neq i}}^{m-1} [d(X_k, X_\ell)]^2 \right\}^{1/2}, \quad \text{for} \quad 0 \leqslant i \leqslant (m - 1).$$

where the K_i^{m-2} is the barycenter of the i-th face of S^{m-1}.

Suppose now that S^m is any m-simplex in $\mathbb{R}^n$, let X_i and X_j, $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant m$, $i \neq j$ be two vertices of S^m and let T_i^{m-1}, K_i^{m-1}, T_{ij}^{m-2}, K_{ij}^{m-2} be as in Notation 3.4. Consider the 2-simplex $< X_i, X_j, K_{ij}^{m-2} >$ as shown in Fig 2.

Figure 2.



Then from Lemma 3.6 it is apparent that

$$d(X_j, K_i^{m-1}) = \frac{m-1}{m} \, d(X_j, K_{ij}^{m-2}),$$

From the above relationship using Lemma 3.7, we get

$$\text{(3.10)} \quad [d(X_i, K_i^{m-1})]^2 = \frac{1}{m} [d(X_i, X_j)]^2 + \frac{m-1}{m} [d(X_i, K_{ij}^{m-2})]^2 - \frac{m-1}{m^2} [d(X_j, K_{ij}^{m-2})]^2$$

Now, by the inductive hypothesis and since

$$T_j^{m-1} = < X_0, X_1, \ldots, X_{j-1}, X_{j-1}, \ldots, X_m >$$

is an $(m-1)$ -simplex in $\mathbb{R}^n$, we obtain using (3.9) that

$$\text{(3.11)} \quad [d(X_i, K_{ij}^{m-2})]^2 = \frac{1}{m-1} \sum_{\substack{l=0 \\ l \neq i,j}}^{m} [d(X_i, X_l)]^2 - \frac{1}{(m-1)^2} \sum_{\substack{k=0 \\ k \neq i,j}}^{m-1} \sum_{\substack{l=k+1 \\ l \neq i,j}}^{m} [d(X_k, X_l)]^2.$$

Similarly, from the $(m-1)$ -simplex $T_i^{m-1} = < X_0, X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_m >$ we obtain that

$$(3.12)\quad [d(X_i, K_{ij}^{m-2})]^2 = \frac{1}{m-1} \sum_{\substack{t=0\\ t\neq i,j}}^{m} [d(X_i, X_t)]^2 - \frac{1}{(m-1)^2} \sum_{\substack{k=0\\ k\neq i,j}}^{m-1} \sum_{\substack{t=\kappa+1\\ t\neq i,j}}^{m} [d(X_k, X_t)]^2$$

Combining (3.10), (3.11) and (3.12) we get

$$[d(X_i, K_i^{m-1})]^2 = \frac{1}{m}[d(X_i, X_j)]^2 + \frac{1}{m}\sum_{\substack{t=0\\ t\neq i,j}}^{m} [d(X_i, X_t)]^2$$

$$- \frac{1}{m(m-1)} \sum_{\substack{k=0\\ k\neq i,j}}^{m-1} \sum_{\substack{t=\kappa+1\\ t\neq i,j}}^{m} [d(X_k, X_t)]^2$$

$$- \frac{1}{m^2}\sum_{\substack{t=0\\ t\neq i,j}}^{m} [d(X_j, X_t)]^2 + \frac{1}{m^2(m-1)} \sum_{\substack{k=0\\ k\neq i,j}}^{m-1} \sum_{\substack{t=\kappa+1\\ t\neq i,j}}^{m} [d(X_k, X_t)]^2$$

or

$$[d(X_i, K_i^{m-1})]^2 = \frac{1}{m}\sum_{\substack{t=0\\ t\neq i}}^{m} [d(X_i, X_t)]^2 - \frac{1}{m^2}\sum_{\substack{t=0\\ t\neq i,j}}^{m} [d(X_j, X_t)]^2$$

$$+ \frac{1-m}{m^2(m-1)} \sum_{\substack{k=0\\ k\neq i,j}}^{m-1} \sum_{\substack{t=\kappa+1\\ t\neq i,j}}^{m} [d(X_k, X_t)]^2$$

or

$$(3.13)\quad [d(X_i, K_i^{m-1})]^2 = \frac{1}{m}\sum_{\substack{t=0\\ t\neq i}}^{m} [d(X_i, X_t)]^2 - \frac{1}{m^2} \sum_{\substack{k=0\\ k\neq i}}^{m-1} \sum_{\substack{t=\kappa+1\\ t\neq i}}^{m} [d(X_k, X_t)]^2$$

Now from (3.13) using Lemma 3.5 we obtain

$$d(X_i, K^m) = \frac{m}{m+1}\left\{ \frac{1}{m}\sum_{\substack{t=0\\ t\neq i}}^{m} [d(X_i, X_t)]^2 - \frac{1}{m^2} \sum_{\substack{k=0\\ k\neq i}}^{m-1} \sum_{\substack{t=\kappa+1\\ t\neq i}}^{m} [d(X_k, X_t]^2 \right\}^{1/2}$$

which completes the proof of the theorem. □

Remark 3.9. Using Theorem 3.8, it is easy to determine the radius of an m-simplex in $\mathbb{R}^n$.

Corollary 3.10. *Let S^m and K^m be as in Theorem 3.8 and let D and M be the diameter and the microdiameter of S^m, respectively. Then for any point T in S^m it is true that*

$$d(T, K^m) \leqslant \frac{m}{m+1} \left\{ D^2 - \frac{m-1}{2m} M^2 \right\}^{1/2} \tag{3.14}$$

Proof. Consider the length Q of the radius A^m of S^m. Then, we have that

$$Q = \max_{0 \leqslant i \leqslant m} \{d(X_i, K^m)\}$$

Using Theorem 3.8, we find, apparently, that

$$Q \leqslant \frac{m}{m+1} \left\{ D^2 - \left[\binom{m+1}{2} - m \right] M^2 / m^2 \right\}^{1/2}$$

or

$$Q \leqslant \frac{m}{m+1} \left\{ D^2 - \frac{m-1}{2m} M^2 \right\}^{1/2}$$

Consider now the closed n-ball B (K^m, Q) with center K^m and radius Q. Then it is easy to see that $S^m \subset B(K^m, Q)$. Consequently, for any point T in S^m we obtain

$$d(T, K^m) \leqslant Q \leqslant \frac{m}{m+1} \left\{ D^2 - \frac{m-1}{2m} M^2 \right\}^{1/2}$$

which proves the corollary. □

Theorem 3.11. Kearfott [6]). *Let S_o^m be an m-simplex in $\mathbb{R}^n$, let p be any positive integer, and let S_p^m be any m-simplex produced after p bisections of S_o^m. Then*

$$D_p \leqslant (\sqrt{3}/2)^{\lfloor p/m \rfloor} D_o \tag{3.15}$$

where D_p and D_o are the diameters of S_p^m and S_o^m, respectively, and $\lfloor p/m \rfloor$ is the largest integer less than or equal to p / m.

Proof. See [6. pp. 1149-1151]

Corollary 3.12. *Let S_o^m, S_p^m, D_o, and D_p be as in Theorem 3.11 and let K_p^m and M_p be the barycenter and the microdiameter of S_p^m, respectively. Then for any point T in S_p^m the following is valid*

$$(3.16) \qquad d(T, K_p^m) \leqslant \frac{m}{m+1} \ (\sqrt{3}/2)^{\lfloor p/m \rfloor} D_o.$$

Proof. By combining (3.14) and (3.15), it follows that

$$d(T, K^m) \leqslant \frac{m}{m+1} \left\{ D_p^2 - \frac{m-1}{2m} M_p^2 \right\}^{1/2} \leqslant \frac{m}{m+1} \ (\sqrt{3}/2)^{\lfloor p/m \rfloor} D_o$$

which proves the corollary. □

Remark 3.13. Better estimates of the bound in (3.16) for triangles can be proved using the results of [1], [12] or [14] instead of Theorem 3.11.

Now we shall apply the preceding results to obtain a proof of convergence and an upper error bound for the generalized method of bisection applied to root-finding.

Definition 3.14. *Let S^n be an n-simplex in $\mathbb{R}^n$; let D and M be the diameter and the microdiameter of S^n, respectively. Suppose that there is a root r of the system (1.1) in S^n. Then we define the barycenter K^n of S^n as an approximation to r and the quantity $E = \frac{n}{n+1} \left\{ D^2 - \frac{n-1}{2n} M^2 \right\}^{1/2}$ as an error estimate for K^n.*

Corollary 3.15. *Suppose that S_p^n is the selected n-simplex produced after p bisections of an n-simplex S_o^n in $\mathbb{R}^n$; let D_p and M_p be the diameter and the microdiameter of S_p^n, respectively. Suppose that r is a solution of the system (1.1), which is included in S_p^n and that K_p^n and E_p are the approximation of r and the error estimate of K_p^n, respectively. Then it can be shown that*

$$(a) \ E_p \leqslant \frac{n}{n+1} (\sqrt{3}/2)^{\lfloor p/n \rfloor} D_0$$

$$(b) \ E_p \leqslant (\sqrt{3}/2)^{\lfloor p/n \rfloor} E_0$$

Proof. By assumption, using (3.16), we have

$$E_p = \frac{n}{n+1}\left\{ D_p^2 - \frac{n-1}{2n} M_p^2 \right\}^{1/2} \leqslant \frac{n}{n+1}(\sqrt{3}/2)^{\lfloor p/n \rfloor} D_0,$$

which proves the first part of the corollary.

Next, using (3.16), we find

$$E_p = \frac{n}{n+1}\left\{ D_p^2 - \frac{n-1}{2n} M_p^2 \right\}^{1/2} \leqslant \frac{n}{n+1}\left\{ (\sqrt{3}/2)^{2\lfloor p/n \rfloor} D_0^2 - \frac{n-1}{2n} M_p^2 \right\}^{1/2}$$

Since $0 \leqslant (\sqrt{3}/2)^{2\lfloor p/n \rfloor} \leqslant 1$ we obtain

$$E_p \leqslant (\sqrt{3}/2)^{\lfloor p/n \rfloor} \frac{n}{n+1}\left\{ D_0^2 - \frac{n-1}{2n} M_0^2 \right\}^{1/2} = (\sqrt{3}/2)^{\lfloor p/n \rfloor} E_0$$

which proves the part (b) of the corollary. □

Corollary 3.16. *Suppose that r, K_p^n and E_p are as in the preceding corollary. Then*

$$E_p \longrightarrow 0 \quad \text{as} \quad p \longrightarrow \infty$$

and

$$K_p^n \longrightarrow r \quad \text{as} \quad p \longrightarrow \infty$$

Proof. It follows directly from the above corollary. □

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