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ON PERTURBATION OF ROOTS OF HOMOGENEOUS ALGEBRAIC SYSTEMS

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Abstract. A problem concerning the perturbation of roots of a system of homogeneous algebraic equations is investigated. The question of conservation and decomposition of a multiple root into simple roots are discussed. The main theorem on the conservation of the number of roots of a deformed (not necessarily homogeneous) algebraic system is proved by making use of a homotopy connecting initial roots of the given system and roots of a perturbed system. Hereby we give an estimate on the size of perturbation that does not affect the number of roots. Further on we state the existence of a slightly deformed system that has the same number of real zeros as the original system in taking the multiplicities into account. We give also a result about the decomposition of multiple real roots into simple real roots.

1. Introduction

The central subject of this paper is an investigation on the perturbation of roots of the system of algebraic equations. Our central Theorem 4.4 states that the number of simple real roots of a system located in a compact set does not change after a sufficiently small perturbation of the system.

As a matter of fact, this kind of fact has been well known to those who study the deformation of the singularities of differentiable mappings. It is, however, a nontrivial question how small this perturbation shall be so that the number of simple real roots in a given compact set remains unchanged. All the existing theorems (see [2] §12.6) do not specify the size of the compact set and the perturbation of the system under question. They simply state that for a compact set and perturbation, both of them small enough, the invariance of the number of roots holds. This situation can be explained by the fact that they simply treat the notion of local algebra, and consequently they are valid only in the germ sense. Here we try to give an estimate on the size of admissible perturbation of the system for a fixed compact set.

Furthermore, we give a result about the decomposition of multiple real roots into simple real roots. In particular, our Theorem 4.7 introduces a condition on the conservation of the number of simple real roots inside of a given ball. This theorem is proved by means of an application of Cauchy–Kovalevskaya's theorem on the quasi-linear differential equation. In addition, Theorem 5.2 gives a criterion on the existence of multiple roots of a system in terms of the zero locus of the Jacobian

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function. At last, Theorem 5.4 assures us the existence of a deformed system of the original system that possesses only simple roots. This fact corresponds to the classical Bertini–Sard theorem on the codimension of the discriminant set corresponding to a system of algebraic equations.

The paper is organized as follows. In Section 2 background material is given. In Section 3 preliminary results on the relation between the basis element of the kth power of the maximal ideal and the gradient vector of a polynomial are presented. This result will be used to prove (in Section 4) the existence of homotopy connecting initial roots of the given system and roots of perturbed system. In Section 4 we state the existence of a slightly deformed system that has the same number of real zeros as the original system in taking the multiplicities into account. In Section 5 we give a result about the decomposition of multiple real roots into simple real roots. The paper ends in Section 6 with some concluding remarks.

2. Background material

Let us first introduce the notation used in the paper.

Throughout the paper, polynomials, ideals, etc. are considered in the framework of the polynomial ring $\mathbb{R}[x]$. We consider the following system of algebraic equations with real coefficients $a_\alpha^{(i)}$:

$$\left\{\begin{aligned} &f_1(x_1, x_2, \ldots, x_n) = 0,\\ &f_2(x_1, x_2, \ldots, x_n) = 0,\\ &\qquad\qquad\vdots\\ &f_n(x_1, x_2, \ldots, x_n) = 0, \end{aligned}\right. \tag{2.1}$$

with

$$\begin{aligned} f_i(x_1, x_2, \ldots, x_n) &= \sum_{|\alpha|=m_i} a_\alpha^{(i)} x^\alpha + a_0^{(i)},\\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad a_0^{(i)} \in \mathbb{R},\\ |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n. \end{aligned} \tag{2.2}$$

Here we recall the notion of the *maximal ideal* $\mathfrak{m}$ of the ring $\mathbb{R}[x]$. It is an ideal generated by the monomials $x_1, \ldots, x_n$, i.e.,

$$\mathfrak{m} = \langle x_1, \ldots, x_n \rangle = \big\{ x_1 g_1(x) + \cdots + x_n g_n(x);\ g_1(x), \ldots, g_n(x) \in \mathbb{R}[x] \big\},$$

which can be identified with an infinite-dimensional set of polynomials with the form

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha,$$

with $\alpha_i \geqslant 1$ for some $i \in [1, n]$. Here $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}^n$ denotes the set of n-tuples of natural numbers. We recall the notation that

$$\text{supp}\ \varphi = \{\alpha \in \mathbb{N}^n;\ \varphi_\alpha \neq 0\},$$

for a polynomial $\varphi(x) = \sum_{\alpha \in \mathbb{N}^n} \varphi_\alpha x^\alpha$. For the polynomial as above one can define so called *Newton diagram.* That is to say the Newton diagram $N(\varphi)$ is a noncompact set contained in $\mathbb{R}^n_{\geqslant 0}$ defined as follows,

$$N(\varphi) = \text{convex hull of} \left\{ \bigcup_{\alpha \in \mathrm{supp}(\varphi)} \{\alpha + \mathbb{R}^n_{\geqslant 0}\} \right\}.$$

Here we used the notation $\mathbb{R}_{\geqslant 0} := \{x \in \mathbb{R};\, x \geqslant 0\}$, the set of nonnegative real numbers and $\mathbb{R}^n_{\geqslant 0}$ the set of n-tuples nonnegative real numbers. In the proof of the main Theorem 4.4, we utilize the construction of a *homotopy* that connects roots of the initial system with corresponding roots of system perturbed by terms with coefficient size $t > 0$. Here, by the notion of homotopy we mean the existence of a smooth curve $x_j(\tau)$ that depends on the parameter $0 \leqslant \tau \leqslant t$ associated to each root x_j of the initial system such that $x_j = x_j(0)$ while $x_j(t)$ is the corresponding root of the perturbed system.

3. Preliminary results

The main result of this section, Proposition 3.4, states a precise way to describe the decomposition of each basis element of the maximal ideal $\mathfrak{m}^k$ into a linear sum of the components of the gradient vector ∇f_ℓ for some ℓ under the condition (3.1) mentioned below.

We repeat the notation as in the preceding section. Further we impose the following condition on the polynomials of (2.1) in such a way that their degrees are ordered as follows:

$$m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n.$$

Let us consider the situation where the gradient ideal $\left\langle \frac{\partial}{\partial x_1} f_\ell, \ldots, \frac{\partial}{\partial x_n} f_\ell \right\rangle$ over $\mathbb{R}[x]$ contains certain power of maximal ideal $\mathfrak{m}^k$. That is to say,

$$\left\langle \frac{\partial f_\ell(x)}{\partial x_1}, \frac{\partial f_\ell(x)}{\partial x_2}, \ldots, \frac{\partial f_\ell(x)}{\partial x_n} \right\rangle \supset \langle x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \rangle_{\alpha_1 + \cdots + \alpha_n \geqslant k}. \tag{3.1}$$

Let us denote by $M_\alpha(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n = k$. It is possible to consider the set of such monomials as a basis of $\mathfrak{m}^k$. The dimension $\mu_n(k)$ of the basis of the ideal $\mathfrak{m}^k$ can be calculated by the recurrence relation

$$\begin{aligned}
\mu_n(k) &= \sum_{i=0}^{k} \mu_{n-1}(k-i), \\
\mu_2(k) &= k+1, \\
\mu_3(k) &= \frac{(k+2)(k+1)}{2}, \\
&\vdots
\end{aligned}$$

Evidently, $\mu_n(k)$ is the number of the entire lattice points on an $(n-1)$-dimensional face of the n–simplex

$$\mu_n(k) = \#\left\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0} : \alpha_1 + \cdots + \alpha_n = k\right\}.$$

Here and henceforth $\mathbb{Z}_{\geqslant 0} := \{m \in \mathbb{Z};\, m \geqslant 0\}$ denotes the set of nonnegative integers and $\mathbb{Z}^n_{\geqslant 0}$ denotes the set of n-tuples of nonnegative integers. For the system

(2.1) with $f_\ell(x)$ satisfying the condition (3.1), we will make use of the notation $\mu := \mu_n(k)$.

We call a germ $\varphi(x)$ convenient at zero when the Newton diagram $N(\varphi)$ of it at zero contains noncompact part of all coordinate axes (cf. [2]). In other words $\varphi(x)$ is convenient at zero if it admits the representation

$$\varphi(x) = \sum_{i=1}^{n} x_i^{\beta_i} + R(x)$$

for $\beta_i \geqslant 1$ and a certain polynomial $R(x)$. It is easy to see that if $f_\ell(x)$ has a convenient germ at zero, then there exists $k \geqslant 1$ such that the condition (3.1) is satisfied.

Suppose that a polynomial vector

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} = F \cdot \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where F is an invertible constant matrix and $\varphi(x) \in \mathfrak{m}^{k+1}$ and $\deg \varphi(x) = k' \geqslant k+1$, i.e., $\operatorname{supp} \varphi \subseteq \{\alpha \in \mathbb{Z}^n_{\geqslant 0};\ k+1 \leqslant |\alpha| \leqslant k'\}$. As the left-hand side of the above relation depends only on the first column of F, the other columns of it do not concern the further argument. Nonetheless we keep this notation for the sake of convenience to prove our main theorem (see (4.3), (4.4), (4.9) below). The question we pose concerns the behavior of roots of a system

$$\begin{cases} f_1(x) + t\,\varphi_1(x) = 0, \\ f_2(x) + t\,\varphi_2(x) = 0, \\ \qquad\vdots \\ f_n(x) + t\,\varphi_n(x) = 0, \end{cases} \tag{3.2}$$

with $t \in [0,1] \subset \mathbb{R}$ as a parameter.

To formulate further statements in a proper way, we introduce the notation

$$f_i(x) = a_1^{(i)} x^{\vec{v}_1^{(i)}} + a_2^{(i)} x^{\vec{v}_2^{(i)}} + \cdots + a_{\lambda_i}^{(i)} x^{\vec{v}_{\lambda_i}^{(i)}},$$

where the vectors $\vec{v}_j^{(i)} = \left(v_{j,1}^{(i)}, v_{j,2}^{(i)}, \ldots, v_{j,n}^{(i)}\right)$, $1 \leqslant j \leqslant \lambda_i$, satisfy

$$\left\langle (1,\ldots,1), \vec{v}_1^{(i)} \right\rangle = \left\langle (1,\ldots,1), \vec{v}_2^{(i)} \right\rangle = \cdots = \left\langle (1,\ldots,1), \vec{v}_{\lambda_i}^{(i)} \right\rangle = m_i.$$

In general, it is not easy to formulate a sufficient condition on $f_\ell(x)$ so that the condition (3.1) holds. Here we propose a simple necessary condition for that.

Proposition 3.1. *The following isomorphism* (3.3) *is necessary so that the condition* (3.1) *holds,*

$$\begin{aligned} &\left\{ b_2 \left(\vec{v}_1^{(\ell)} - \vec{v}_2^{(\ell)} \right) + \cdots + b_\ell \left(\vec{v}_1^{(\ell)} - \vec{v}_{\lambda_\ell}^{(\ell)} \right);\ (b_2, \ldots, b_\ell) \in \mathbb{Z}^{\ell-1} \right\} \\ &\qquad\qquad \cong \left\{ \vec{\alpha} \in \mathbb{Z}^n;\ \langle (1,\ldots,1), \vec{\alpha} \rangle = 0 \right\}. \end{aligned} \tag{3.3}$$

Proof. If the condition (3.3) does not hold, it is evidently impossible to create all monomials x^α with $|\alpha| = k$ as a linear combination of $\frac{\partial}{\partial x_i} f_\ell(x)$'s. $\square$

Remark 3.2. This isomorphism can be realized by shifting each lattice point $\vec{v}$ of the right-hand side of relation (3.3) toward the lattice point $\vec{v} + (m_i, 0, \ldots, 0)$.

We give now an example for which condition (3.1) does not hold.

Example 3.3. Let us consider the system

$$\begin{cases} f_1(x_1, x_2) = a_1^{(1)} x_1^6 + a_2^{(1)} x_1^3 x_2^3 + a_3^{(1)} x_2^6 = 0, \\ f_2(x_1, x_2) = a_1^{(2)} x_1^{12} + a_2^{(2)} x_1^6 x_2^6 + a_3^{(2)} x_2^{12} = 0, \end{cases} \tag{3.4}$$

where

$$\left(a_2^{(i)}\right)^2 - 4 a_1^{(i)} a_3^{(i)} \neq 0, \quad i = 1, 2.$$

For these polynomials the lattice defined on the left-hand side of relation (3.3) is isomorphic to

$$\left\{ \vec{\alpha} \in \mathbb{Z}^2; \;\; \langle (1,1), \vec{\alpha} \rangle = 0; \;\; \frac{\vec{\alpha}}{3} \in \mathbb{Z}^2 \right\}.$$

Thus, in this case, the condition (3.1) does not hold.

From now on, we use the notation $f_\ell(x, \mathbf{a})$ instead of $f_\ell(x)$ if we want to emphasize its dependence on the coefficients $\mathbf{a} = (a_1^\ell, \ldots, a_{\lambda_\ell}^\ell)$. For a set of polynomials $\Lambda_1^r(x, \mathbf{a}), \Lambda_2^r(x, \mathbf{a}), \ldots, \Lambda_{\nu_r}^r(x, \mathbf{a}) \in \mathbb{R}[x, \mathbf{a}]$ homogeneous in variables $\mathbf{a}$ we consider the linear combinations

$$\Lambda_j^{r+1}(x, \mathbf{a}) = \sum_{i=1}^{\nu_0} \gamma_i^{(j)}(\mathbf{a})\, x^{\vec{\beta}_i^{(j)}}\, \Lambda_i^r(x, \mathbf{a}), \quad j = 1, 2, \ldots, \nu_{r+1},$$

where $\vec{\beta}_i^{(j)} \in \mathbb{Z}_{\geqslant 0}^n$ and $\gamma_i^{(j)}(\mathbf{a})$ are linear polynomials in variables $\mathbf{a}$.

Proposition 3.4. *Let us consider the chain of polynomials sets*

$$\left\{ \Lambda_1^r(x, \mathbf{a}), \Lambda_2^r(x, \mathbf{a}), \ldots, \Lambda_{\nu_r}^r(x, \mathbf{a}) \right\}, \quad r = 0, 1, 2, \ldots,$$

as above with

$$\Lambda_1^0(x, \mathbf{a}) = \frac{\partial}{\partial x_1} f_\ell(x, \mathbf{a}), \ldots, \Lambda_n^0(x, \mathbf{a}) = \frac{\partial}{\partial x_n} f_\ell(x, \mathbf{a}).$$

Suppose that for certain $r = L$, some of the $\Lambda_*^L(x, \mathbf{a})$'s coincide with $\lambda_\ell^{(s)}(\mathbf{a}) M_s(x)$. That is to say there exists $\bar{h}_{1,\ell}^{(s)}(x, \mathbf{a}), \ldots, \bar{h}_{n,\ell}^{(s)}(x, \mathbf{a}) \in \mathbb{R}[x, \mathbf{a}]$ such that

$$\lambda_\ell^{(s)}(\mathbf{a}) M_s(x) = \sum_{i=1}^{n} \bar{h}_{i,\ell}^{(s)}(x, \mathbf{a}) \frac{\partial}{\partial x_i} f_\ell(x, \mathbf{a}).$$

Then $\deg_{\mathbf{a}} \lambda_\ell^{(s)}(\mathbf{a}) = L$ *and* $\deg_{\mathbf{a}} \bar{h}_{i,\ell}^{(s)}(x, \mathbf{a}) = L - 1$.

Proof. After the definition of the recursive process to create $\Lambda_j^{r+1}(x, \mathbf{a})$ from $\Lambda_j^r(x, \mathbf{a})$ it is clear that $\Lambda_j^r(x, \mathbf{a})$ is a homogeneous polynomial of degree $r+1$ in $\mathbf{a}$. The statement is the direct consequence of this fact. $\square$

Example 3.5. We consider the example

$$f_\ell(x_1, x_2, \mathbf{a}) = f(x, \mathbf{a}) = a_1x_1^5 + a_2x_1^2x_2^3 + a_3x_2^5 = 0,$$

with

$$\frac{\partial}{\partial x_1} f(x, \mathbf{a}) = 5a_1x_1^4 + 2a_2x_1x_2^3,$$
$$\frac{\partial}{\partial x_2} f(x, \mathbf{a}) = 3a_2x_1^2x_2^2 + 5a_3x_2^4.$$

Then we have the following chain of polynomials to get $x^{\vec{\alpha}_2} = x_1^2x_2^8$ as a linear combination of $\theta_1 = \frac{\partial}{\partial x_1} f(x, \mathbf{a})$ and $\theta_2 = \frac{\partial}{\partial x_2} f(x, \mathbf{a})$:

$$\Lambda_1^1(x, \mathbf{a}) := (2a_2x_1x_2^5 - 5a_1x_1^4x_2^2)\,\theta_1,$$
$$\Lambda_2^1(x, \mathbf{a}) := (3a_2x_1^6 - 5a_3x_1^4x_2^2)\,\theta_2,$$
$$\Lambda_3^1(x, \mathbf{a}) := 5a_3x_1^2x_2^4\,\theta_2,$$

$$\Lambda_1^2(x, \mathbf{a}) := 3a_2\Lambda_1^1(x, \mathbf{a}),$$
$$\Lambda_2^2(x, \mathbf{a}) := 3a_2\Lambda_2^1(x, \mathbf{a}) + 5a_3\Lambda_3^1(x, \mathbf{a}),$$

$$\Lambda_1^3(x, \mathbf{a}) := 3a_2\Lambda_1^2(x, \mathbf{a}),$$
$$\Lambda_2^3(x, \mathbf{a}) := 5a_1\Lambda_2^2(x, \mathbf{a}),$$

$$\Lambda_1^4(x, \mathbf{a}) := 3a_2\Lambda_1^3(x, \mathbf{a}) + 5a_1\Lambda_2^3(x, \mathbf{a}) = (2^2\ 3^3a_2^5 + 5^5a_1^2a_3^3)x_1^2x_2^8.$$

Thus we have

$$\lambda^{(2)}(\mathbf{a}) = 2^2\ 3^3a_2^5 + 5^5a_1^2a_3^3,$$
$$\bar{h}_1^{(2)}(x, \mathbf{a}) = (3a_2)^3(2a_2x_1x_2^5 - 5a_1x_1^4x_2^2),$$
$$\bar{h}_2^{(2)}(x, \mathbf{a}) = (5a_1)^2((3a_2)^2x_1^6 - 15a_2a_3x_1^4x_2^2 - (5a_3)^2x_1^2x_2^4).$$

For the case of $x^{\vec{\alpha}_5} = x_1^5x_2^5$, we have

$$\lambda^{(5)}(\mathbf{a}) = 5a_1(2^2 3^3a_2^5 + 5^5a_1^2a_3^3),$$
$$\bar{h}_1^{(5)}(x, \mathbf{a}) = 5a_1^2a_2^3x_1x_2^5 + 2 \cdot 3^3\ 5a_1a_2^4x_1^4x_2^2,$$
$$\bar{h}_2^{(5)}(x, \mathbf{a}) = -2 \cdot 5^2a_1^2a_2((3a_2)^2x_1^6 - 15a_2a_3x_1^4x_2^2 - (5a_3)^2x_1^2x_2^4).$$

For the case of $x^{\vec{\alpha}_{10}} = x_2^{10}$, we have

$$\lambda^{(10)}(\mathbf{a}) = 5a_3(2^2 3^3a_2^5 + 5^5a_1^2a_3^3),$$
$$\bar{h}_1^{(10)}(x, \mathbf{a}) = (2^2 3^3a_2^5 + 5^5a_1^2a_3^3)x_2^6 \\ -3a_2(5a_1)^2((3a_2)^2x_1^6 - 15a_2a_3x_1^4x_2^2 - (5a_3)^2x_1^2x_2^4),$$
$$\bar{h}_2^{(10)}(x, \mathbf{a}) = -(3a_2)^4(2a_2x_1x_2^5 - 5a_1x_1^4x_2^2).$$

4. The number of roots of a deformed system

In this section we state that a slightly deformed system has the same number of zeros as the original system in taking the multiplicities into account. We recall here that the index $\ell \in [1, n]$ has been fixed so that $f_\ell(x)$ satisfies the condition (3.1).

Under the assumption that supp $\varphi \subseteq \{\alpha \in \mathbb{Z}_{\geqslant 0}^n;\ k+1 \leqslant |\alpha| \leqslant k'\}$, we can define the following norms.

Definition 4.1. We introduce the norm

$$\|\varphi_\ell\| = \sum_{\alpha \in \operatorname{supp} \varphi_\ell} |\alpha|\, |\varphi_{\ell,\alpha}|,$$

where $\varphi_\ell(x) = \sum_{\alpha \in \operatorname{supp} \varphi_\ell} \varphi_{\ell,\alpha} x^\alpha$. For each compact set $\mathbf{K}$, we define the value

$$C_{\mathbf{K}}(\mathbf{a}) = \max_{1 \leqslant s \leqslant \mu} \left(\sum_{1 \leqslant j \leqslant n} \sum_{|\vec{\beta}| \leqslant k'-k-1} \max_{x \in \mathbf{K}} \left| h_{j,\ell}^{(s)}(x,\mathbf{a}) x^{\vec{\beta}} \right| \right),$$

where $h_{j,\ell}^{(s)}(x,\mathbf{a}) = \frac{\bar{h}_{j,\ell}^{(s)}(x,\mathbf{a})}{\lambda_\ell^{(s)}(\mathbf{a})}$ after the notation of Proposition 3.4. $\square$

Remark 4.2. In general we cannot give any reasonable estimate on $C_{\mathbf{K}}(\mathbf{a})$. In Example 3.5, $h_{j,\ell}^{(s)}(x,\mathbf{a})$ contains coefficients of the form

$$\frac{\text{polynomial of degree 5 in } (a_1, a_2, a_3)}{a_3(2^2\, 3^3\, a_2^5 + 5^5\, a_1^2\, a_3^3)}.$$

This value can be as large as possible if the denominator is very near to zero. The coefficients of $h_{s,i}^j(x,\mathbf{a})$ contain rational functions in the variable $\mathbf{a}$, with denominators $\lambda^{(s)}(\mathbf{a})$ introduced in Proposition 3.4. $\square$

Before formulating our main theorem, we recall a simple lemma of linear algebra. The notation id_μ stands for the indentity matrix of size μ.

Lemma 4.3. *Let us consider $\mu \times \mu$ real matrix $A = (a_{ij})$. If $|a_{ij}| < \frac{1}{\mu^2}$, then $(\mathrm{id}_\mu + A)$ is invertible.*

Proof. By straightforward calculation of the determinant of $(\mathrm{id}_\mu + A)$ we have

$$\det(\mathrm{id}_\mu + A) = 1 + a_{11} + a_{22} + \cdots + a_{\mu\mu} + R(a),$$

where $R(a)$ is a polynomial containing $(\mu^2 - \mu - 1)$ terms of monomials in (a_{ij}) whose degrees are higher than or equal to 2 and less than or equal to μ. Evidently, under the condition $|a_{ij}| < \frac{1}{\mu^2}$ we obtain that $\det(\mathrm{id}_\mu + A) \neq 0$. $\square$

Theorem 4.4. *The number of simple roots of the system* (3.2) *inside of a compact set* $\mathbf{K}$ *coincides with that of the system* (2.1) *if t satisfies the inequality*

$$t < \frac{1}{\|\varphi_\ell\|\, C_{\mathbf{K}}(\mathbf{a})\, \mu^2}. \tag{4.1}$$

Proof. Our strategy consists of the construction of a homotopy that connects the simple roots of system (2.1) and those of (3.2).

Suppose that we succeed in constructing a homotopy $x(\tau)$, $0 \leqslant \tau \leqslant t$, with $x(0) = x$ such that

$$f_s(x(\tau)) + \tau \varphi_s(x(\tau)) = f_s(x), \quad 1 \leqslant s \leqslant n,$$

then the vector field along it satisfies the equality

$$\frac{d}{d\tau} \begin{pmatrix} f_1(x(\tau)) \\ f_2(x(\tau)) \\ \vdots \\ f_n(x(\tau)) \end{pmatrix} = \sum_{i=1}^n \frac{dx_i}{d\tau}(\tau) \frac{\partial}{\partial x_i} \begin{pmatrix} f_1(x(\tau)) \\ f_2(x(\tau)) \\ \vdots \\ f_n(x(\tau)) \end{pmatrix}.$$

In applying this relation to system (3.2), we get

$$\sum_{i=1}^{n} \dot{x}_i(\tau)\frac{\partial}{\partial x_i}\left\{\begin{pmatrix} f_1(x(\tau)) \\ f_2(x(\tau)) \\ \vdots \\ f_n(x(\tau)) \end{pmatrix} + \tau F \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} + F \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Further, we shall realize a smooth homotopy

$$\varphi_\ell(x) = \left(\sum_{i=1}^{n} v_i(x,\tau)\frac{\partial}{\partial x_i}\right)(f_\ell\,(x) + \tau\varphi_\ell\,(x)).$$

We remember that we denoted the basis of $\mathfrak{m}^k$ by $M_{\alpha_i}(x)$, $1 \leqslant i \leqslant \mu = \mu_n(k)$. The condition (3.1) entails the relation

$$\begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix} = H^{(1)}\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \end{pmatrix} + H^{(2)}\begin{pmatrix} \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} \\ \vdots \\ \frac{\partial f_n}{\partial x_2} \end{pmatrix} + \cdots + H^{(n)}\begin{pmatrix} \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_n} \end{pmatrix}, \tag{4.2}$$

for some polynomial entry rank–1 $(\mu \times n)$ matrices $H^{(1)}, H^{(2)}, \ldots, H^{(n)}$ of the form

$$H^{(i)} = \mu\left\{\overbrace{\begin{pmatrix} 0 & \cdots & h^{(1)}_{i,\ell} & 0 & \cdots & 0 \\ 0 & \cdots & h^{(2)}_{i,\ell} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & h^{(\mu)}_{i,\ell} & 0 & \cdots & 0 \end{pmatrix}}^{n}\right. ,$$

where $h^{(s)}_{j,\ell}(x,\mathbf{a}) = \frac{\bar{h}^{(s)}_{j,\ell}(x,\mathbf{a})}{\lambda^{(s)}_\ell(\mathbf{a})}$ after the notation of Proposition 3.4 is concentrated at the ℓth column of the matrix $H^{(i)}$. One rewrites the relation (4.2) as

$$\begin{aligned} \begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix} &= \left(\sum_{i=1}^{n} H^{(i)}\frac{\partial}{\partial x_i}\right)\begin{pmatrix} f_1 + \tau\varphi_1 \\ f_2 + \tau\varphi_2 \\ \vdots \\ f_n + \tau\varphi_n \end{pmatrix} \\ &\quad -\tau\left(\sum_{i=1}^{n} H^{(i)}F\frac{\partial}{\partial x_i}\right)\begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \tag{4.3}$$

As we supposed that $\varphi(x) \in \mathfrak{m}^{k+1}$, it is easy to see that

$$\left(\sum_{i=1}^{n} H^{(i)}F\frac{\partial}{\partial x_i}\right)\begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A(x)\begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix}, \tag{4.4}$$

with certain polynomial $(\mu \times \mu)$ matrix $A(x)$, where

$$A(x) = \begin{pmatrix} g_1^{(1)}(x) & \cdots & g_1^{(\mu)}(x) \\ \vdots & \ddots & \vdots \\ g_\mu^{(1)}(x) & \cdots & g_\mu^{(\mu)}(x) \end{pmatrix}.$$

By recalling (4.3), we obtain the equation

$$(\mathrm{id}_\mu + \tau A(x)) \begin{pmatrix} M_{\alpha_1}(x) \\ M_{\alpha_2}(x) \\ \vdots \\ M_{\alpha_\mu}(x) \end{pmatrix} = \left(\sum_{i=1}^{n} H^{(i)} \frac{\partial}{\partial x_i} \right) \begin{pmatrix} f_1 + \tau\varphi_1 \\ f_2 + \tau\varphi_2 \\ \vdots \\ f_n + \tau\varphi_n \end{pmatrix}.$$

Supposing that τ is very small, get the inverse to

$$(\mathrm{id}_\mu + \tau A(x)), \tag{4.5}$$

in the domain $\{x;\, \det(\mathrm{id}_\mu + \tau A(x)) \neq 0\}$. The inequality (4.1) ensures the invertibility of the matrix (4.5). To show this, in view of Lemma 4.3, it is enough to verify that for such a value of τ we have

$$\tau \left(\max_{1 \leqslant i,s \leqslant \mu} \max_{x \in \mathbf{K}} \left| g_s^{(i)}(x) \right| \right) < \frac{1}{\mu^2}. \tag{4.6}$$

In other words, it is enough to prove that

$$\max_{1 \leqslant i,s \leqslant \mu} \max_{x \in \mathbf{K}} \left| g_s^{(i)}(x) \right| < \|\varphi_\ell\| \, C_{\mathbf{K}}(\mathbf{a}). \tag{4.7}$$

We remember that $\operatorname{supp} g_s^{(i)} \subset \{\alpha \in \mathbb{Z}^n;\ k' - m_n \leqslant |\alpha| \leqslant k' - m_i\}$. This is a direct consequence of (4.4). As we have $\operatorname{supp} \varphi_\ell \subset \{\alpha \in \mathbb{Z}^n;\ \ k+1 \leqslant\ |\alpha| \leqslant k'\}$, we can find for every $1 \leqslant \lambda \leqslant n$ a series of polynomials $\xi_{\ell,\lambda}^{(1)}(x), \ldots, \xi_{\ell,\lambda}^{(\mu)}(x)$ such that

$$\frac{\partial}{\partial x_\lambda} \varphi_\ell(x) = \sum_{c=1}^{\mu} \xi_{\ell,\lambda}^{(c)}(x) \, M_c(x). \tag{4.8}$$

In terms of these polynomials,

$$g_s^{(c)}(x) = \sum_{\lambda=1}^{n} h_{\lambda,\ell}^{(s)}(x) \, \xi_{\ell,\lambda}^{(c)}(x),$$

and

$$\operatorname{supp} \xi_{\ell,\lambda}^{(c)} \subset \{\alpha \in \mathbb{Z}^n; 0 \leqslant |\alpha| \leqslant k' - k - 1\}.$$

The absolute value of each coefficient of $\xi^{(c)}_{\ell,\lambda}(x)$ can be estimated by $\|\varphi_\ell\|$ after (4.8) above. We replace $\xi^{(c)}_{\ell,\lambda}(x)$ by $\|\varphi_\ell\| \times \left(\sum_{\vec{\beta}\in \operatorname{supp} \xi^{(c)}_{\ell,\lambda}} x^{\vec{\beta}}\right)$ and we get the inequality

$$\begin{aligned}
&\max_{1\leqslant i,s\leqslant \mu} \max_{x\in\mathbf{K}} \left|g^{(i)}_s(x)\right| \\
&\quad < \left(\sum_{\alpha\in \operatorname{supp}\varphi} |\alpha|\,|\varphi_{\ell,\alpha}|\right) \\
&\quad\quad \times \left(\max_{1\leqslant s\leqslant \mu} \sum_{1\leqslant j\leqslant n} \sum_{\vec{\beta}\in \coprod_{i=1}^{n} \operatorname{supp} \frac{\partial\varphi_\ell}{\partial x_i} \setminus \coprod_{s=1}^{\mu} \operatorname{supp} M_s} \max_{x\in\mathbf{K}} \left|h^{(s)}_{j,\ell}(x,\mathbf{a})\, x^{\vec{\beta}}\right|\right),
\end{aligned}$$

where $A\setminus B = \{\alpha-\beta \in \mathbb{Z}^n_{\geqslant 0};\ \alpha\in A, \beta\in B\}$. the relation (4.8) explains the summand of the above inequality. Therefore, if we set $C_{\mathbf{K}}(\mathbf{a})$ as in Definition 4.1, we obtain the inequality (4.7). Evidently, $C_{\mathbf{K}}(\mathbf{a})$ depends not on the coefficients of $\varphi_1(x),\ldots,\varphi_n(x)$ but on the powers k and k'. This proves the invertibility of the matrix (4.5). Thus,

$$\begin{pmatrix} M_{\alpha_1}(x)\\ M_{\alpha_2}(x)\\ \vdots\\ M_{\alpha_\mu}(x)\end{pmatrix} = (\mathrm{id}_\mu + \tau A(x))^{-1}\left(\sum_{i=1}^{n} H^{(i)}\frac{\partial}{\partial x_i}\right)\begin{pmatrix} f_1+\tau\varphi_1\\ f_2+\tau\varphi_2\\ \vdots\\ f_n+\tau\varphi_n\end{pmatrix}. \tag{4.9}$$

On the other hand,

$$\begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\\ \vdots\\ \varphi_n(x)\end{pmatrix} = F\cdot\begin{pmatrix}\varphi(x)\\ 0\\ \vdots\\ 0\end{pmatrix} = F\cdot G\begin{pmatrix} M_{\alpha_1}(x)\\ M_{\alpha_2}(x)\\ \vdots\\ M_{\alpha_\mu}(x)\end{pmatrix}$$

for some rank–1 $(n\times\mu)$ polynomial matrix

$$G = \begin{pmatrix} g_1(x) & g_2(x) & \cdots & g_\mu(x)\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}.$$

If we apply $F\cdot G$ from the left to the relation (4.9), we get

$$\begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\\ \vdots\\ \varphi_n(x)\end{pmatrix} = F\cdot G(\mathrm{id}_\mu + \tau A(x))^{-1}\left(\sum_{i=1}^{n} H^{(i)}\frac{\partial}{\partial x_i}\right)\begin{pmatrix} f_1+\tau\varphi_1\\ f_2+\tau\varphi_2\\ \vdots\\ f_n+\tau\varphi_n\end{pmatrix}, \tag{4.10}$$

or

$$\begin{pmatrix}\varphi(x)\\ 0\\ \vdots\\ 0\end{pmatrix} = G(\mathrm{id}_\mu + \tau A(x))^{-1}\left(\sum_{i=1}^{n} H^{(i)}\frac{\partial}{\partial x_i}\right)\begin{pmatrix} f_1+\tau\varphi_1\\ f_2+\tau\varphi_2\\ \vdots\\ f_n+\tau\varphi_n\end{pmatrix}. \tag{4.11}$$

This relation gives rise to the equality

$$
(4.12)\qquad \begin{pmatrix} \varphi(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{i=1}^{n} \begin{array}{c} \begin{array}{ccccccc} & & & \ell & & & \\ & & & \Downarrow & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & v_i(x,\tau) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \frac{\partial}{\partial x_i} \begin{pmatrix} f_1 + \tau\varphi_1 \\ f_2 + \tau\varphi_2 \\ \vdots \\ f_n + \tau\varphi_n \end{pmatrix},
$$

where the matrix in front of the derivative has a single nonzero ℓth column. That is to say we obtain the equalities

$$
(4.13)\qquad \begin{aligned} \varphi(x) &= \left(\sum_{i=1}^{n} v_i(x,\tau) \frac{\partial}{\partial x_i} \right) (f_\ell + \tau\varphi_\ell), \\ \varphi_\ell(x) &= F_{\ell 1}\varphi(x). \end{aligned}
$$

The estimate (4.1) ensures that $v_i(x,\tau)$ are real analytic in $x \in \mathbf{K}$. Thus we have constructed a vector field corresponding to the homotopy we need. $\square$

Example 4.5. Let us consider the following system [4]:

$$
(4.14)\qquad \begin{cases} f_1(x_1,x_2) = x_1^2 - x_2^2 - 1 = 0, \\ f_2(x_1,x_2) = x_1^2 + x_2^2 - 2 = 0. \end{cases}
$$

This system has four real solutions within the square $\mathbf{K} = [-2,2]^2$:

$$
(\pm\sqrt{1.5}, \pm\sqrt{0.5}) \approx (\pm 1.22474487139159, \pm 0.70710678118655).
$$

If we perturb this system with a cubic monomial $\varphi(x) = x_1 x_2^2$,

$$
(4.15)\qquad \begin{cases} F_1(x_1,x_2) = x_1^2 - x_2^2 + t x_1 x_2^2 - 1 = 0, \\ F_2(x_1,x_2) = x_1^2 + x_2^2 - 2 = 0, \end{cases}
$$

we calculate the constants $C_{\mathbf{K}}(\mathbf{a}) = \frac{5}{2}$, $\|\varphi\| = 3, \mu = 2$. Therefore we have four solutions of the system (4.15) if $t < \frac{1}{2\cdot 3\cdot 5}$. In particular, if we use the value $t = 0.033 < 1/30$ by applying the rootfinding method of [13, 14, 15], we obtain the following four solutions:

$$
\begin{aligned} &(1.22054232589618, \quad \pm 0.71433635683474), \\ &(-1.22879457180552, \quad \pm 0.70004564158438). \end{aligned}
$$

Example 4.6. Let us consider the system

$$
(4.16)\qquad \begin{cases} f_1(x_1,x_2,x_3) = \left(x_1^2 + \dfrac{x_2^2}{4} - x_3^2 \right) \left(\dfrac{x_1^2}{4} + x_2^2 - x_3^2 \right) - \dfrac{x_3^4}{81} = 0, \\ f_2(x_1,x_2,x_3) = (x_1 + x_2)^2 + 36(x_1 - x_2)^2 - 9x_3^2 = 0, \\ f_3(x_1,x_2,x_3) = \dfrac{x_1^2}{4} + x_2^2 + \dfrac{x_3^2}{9} - 1 = 0. \end{cases}
$$

By applying the rootfinding method of [13, 14, 15] we obtain the following sixteen real solutions within the cube $\mathbf{K} = [-2, 2]^3$:

$$
\begin{array}{l}
(\ \ 0.62830967308983, \ \ 0.91412675198426, \ \pm 0.76883755100759), \\
(-0.62830967308983, -0.91412675198426, \ \pm 0.76883755100759), \\
(\ \ 0.49635596537865, \ \ 0.91441703848857, \ \pm 0.95929271740718), \\
(-0.49635596537865, -0.91441703848857, \ \pm 0.95929271740718), \\
(\ \ 1.11731818404380, \ \ 0.76796989195429, \ \pm 0.93973420474984), \\
(-1.11731818404380, -0.76796989195429, \ \pm 0.93973420474984), \\
(\ \ 1.22450432822695, \ \ 0.66467487192937, \ \pm 1.28459776563576), \\
(-1.22450432822695, -0.66467487192937, \ \pm 1.28459776563576).
\end{array}
$$

We observe that these roots are invariant under the actions of a group $G := (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ generated by two generators $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$ and $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ due to the invariance of the system (4.16) itself under the same group action. If we perturb this system with a quadratic monomial $\varphi(x) = x_2^2$,

$$
(4.17) \qquad \begin{cases}
f_1(x_1, x_2, x_3) = \left(x_1^2 + \dfrac{x_2^2}{4} - x_3^2\right)\left(\dfrac{x_1^2}{4} + x_2^2 - x_3^2\right) - \dfrac{x_3^4}{81} = 0, \\[2ex]
f_2(x_1, x_2, x_3) = (x_1 + x_2)^2 + 36(x_1 - x_2)^2 - 9x_3^2 + t x_2^2 = 0, \\[2ex]
f_3(x_1, x_2, x_3) = \dfrac{x_1^2}{4} + x_2^2 + \dfrac{x_3^2}{9} - 1 = 0,
\end{cases}
$$

we calculate the constants $C_{\mathbf{K}}(\mathbf{a}) = \frac{1}{2}$, $\|\varphi\| = 2, \mu = 3$. Therefore we have sixteen solutions of the system (4.17) if $t < \frac{1}{3^2}$. In particular, if we use the value $t = 0.1 < 1/9$ by applying the rootfinding method of [13, 14, 15] we obtain the following sixteen real solutions within the cube $\mathbf{K} = [-2, 2]^3$:

$$
\begin{array}{l}
(\ \ 0.63087661393950, \ \ 0.91351892559324, \ \pm 0.77060795720733), \\
(-0.63087661393950, -0.91351892559324, \ \pm 0.77060795720733), \\
(\ \ 0.49896002229193, \ \ 0.91405620623649, \ \pm 0.95934810309529), \\
(-0.49896002229193, -0.91405620623649, \ \pm 0.95934810309529), \\
(\ \ 1.11568183127565, \ \ 0.76857206607484, \ \pm 0.93967783245553), \\
(-1.11568183127565, -0.76857206607484, \ \pm 0.93967783245553), \\
(\ \ 1.22357424633595, \ \ 0.66527556809517, \ \pm 1.28379297777855), \\
(-1.22357424633595, -0.66527556809517, \ \pm 1.28379297777855).
\end{array}
$$

One remarks here also that the invariance of the roots under the above mentioned group G are due to the invariance of the system (4.17) itself. $\square$

Before stating a new theorem, let us introduce the norm for a vector function $\vec{v}(x, y) = (v_1(x, y), \ldots, v_n(x, y))$ defined for a pair of values $x, y \in \mathbf{K}$ for some compact set $\mathbf{K}$

$$
\max_{x,y \in \mathbf{K}} \big|\text{vector component of } \vec{v}(x, y)\big| = \max_{1 \leqslant i \leqslant n} \ \max_{x,y \in \mathbf{K}} \big|v_i(x, y)\big|.
$$

As for equation (2.1), we establish the following.

Theorem 4.7. *Let us consider a system of algebraic equations obtained as a perturbation of* (2.1)*:*

$$\left\{\begin{aligned} &F_1(x_1,x_2,\dots,x_n)=0,\\ &F_2(x_1,x_2,\dots,x_n)=0,\\ &\qquad\quad\vdots\\ &F_n(x_1,x_2,\dots,x_n)=0.\end{aligned}\right. \tag{4.18}$$

Suppose that on a ball $\mathbf{B}_r=\{x\in\mathbb{R}^n;\ |x|\leqslant r\}$ *we have*

$$\operatorname{rank}\left(\frac{\partial}{\partial x_j}f_k(x)\right)_{1\leqslant j,k\leqslant n}=n. \tag{4.19}$$

Furthermore, we impose a condition on $(F_1,F_2,\dots,F_n)$*,*

$$\max_{x,y\in\mathbf{B}_r}\left|\textit{vector component of }\left(\frac{\partial}{\partial y_j}f_k(y)\right)^{-1}_{1\leqslant j,k\leqslant n}\begin{bmatrix}(F_1-f_1)(x)\\(F_2-f_2)(x)\\\vdots\\(F_n-f_n)(x)\end{bmatrix}\right|<\varepsilon, \tag{4.20}$$

$$\max_{x,y\in\mathbf{B}_r}\left|\textit{vector component of }\left(\frac{\partial}{\partial y_j}F_k(y)\right)^{-1}_{1\leqslant j,k\leqslant n}\begin{bmatrix}(F_1-f_1)(x)\\(F_2-f_2)(x)\\\vdots\\(F_n-f_n)(x)\end{bmatrix}\right|<\varepsilon, \tag{4.21}$$

where ε *is strictly less than the distance of any root of* (2.1) *in* $\mathbf{B}_r$ *to the boundary* $\partial\,\mathbf{B}_r$*. Suppose that the system* (2.1) *has no multiple real roots. Under these assumptions the equality*

$$\#\{\textit{real simple roots of } (2.1) \textit{ in } \mathbf{B}_r\}=\#\{\textit{real simple roots of } (4.18) \textit{ in } \mathbf{B}_r\},$$

holds.

Proof. We solve the homotopy equation with respect to smooth diffeomorphism

$$x_i+h_i(x,\tau),\qquad 0\leqslant i\leqslant n,$$

that satisfies

$$h_i(x,0)=0, \tag{4.22}$$

$$\begin{aligned}&f_k(x_0+h_0(x,\tau),\dots,x_n+h_n(x,\tau))\\ &\qquad=f_k(x_0,\dots,x_n)+\tau(F_k(x)-f_k(x)),\quad 0\leqslant\tau\leqslant 1.\end{aligned} \tag{4.23}$$

The system gives rise to a system of $(n+1)$ nonlinear differential equations

$$\sum_{j=0}^{n} \frac{\partial h_j}{\partial \tau} \partial_j f_k(x_0 + h_0(x,\tau), \dots, x_n + h_n(x,\tau)) = F_k(x) - f_k(x), \tag{4.24}$$

for $k = 0, 1, \dots, n$. From the assumption (4.19), equation (4.24) is always solvable in the class of real analytic functions so far as

$$\det\left(\left(\frac{\partial}{\partial x_j} f_k(x)\right)_{1 \leqslant j,k \leqslant n}\right) \neq 0,$$

after Cauchy–Kovalevskaya's theorem [5] on the quasi-linear partial differential equation. After the conditions (4.20)–(4.21) and equation (4.24), $|\frac{\partial h_j}{\partial \tau}|$ is always strictly less than ε. Therefore $|h_j(\tau)| < \varepsilon\tau$ and $|x + h_j(\tau)| < |x| + \varepsilon\tau < r$ for x root of (2.1) located in the ball $\mathbf{B}_r$. Thus the homotopy equation admits a real analytic solution that connects $x \in \mathbf{B}_r$ with $x + h(x,1) \in \mathbf{B}_r$. □

Corollary 4.8. *If the conditions on the analyticity of the homotopy constructed in Theorems* 4.4 *and* 4.7 *are fulfilled, then none of the roots of the deformed system encounters another and, consequently, no new multiple roots are created after the proposed perturbation.*

Proof. Assume that after the proposed perturbation a multiple root is created. Then the homotopy constructed in the above Theorems 4.4 and 4.7 looses its analyticity with respect to the parameter τ. □

5. Decomposition of multiple roots

In this section we recall facts about the decomposition of multiple roots into simple roots.

Definition 5.1. For the system (2.1), we define the Jacobian function

$$\operatorname{jac}(f)(x) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}. \tag{5.1}$$

Let us denote by $V_{\operatorname{jac}(f)} = \{x \in \mathbb{R}^n; \operatorname{jac}(f)(x) = 0\}$ the zero set of $\operatorname{jac}(f)(x)$. We use the notation $Q_{\operatorname{jac}(f)}$ for the set

$$V_{\operatorname{jac}(f)} \cap \left\{x \in \mathbb{R}^n;\ \left(f_1^2 + \cdots + f_n^2\right)(x) = 0\right\}.$$

Then we have the following obvious result in view of the definition of $Q_{\operatorname{jac}(f)}$.

Theorem 5.2. *If $Q_{\operatorname{jac}(f)} = \emptyset$, then the system* (2.1) *has only simple real roots, while if $Q_{\operatorname{jac}(f)} \neq \emptyset$, then the system* (2.1) *has multiple real roots.*

Definition 5.3. Let us denote by (Jf) a vector valued ideal

$$(Jf) = \left\langle \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \mathbb{R}[x] \\ \vdots \\ \mathbb{R}[x] \end{bmatrix} \right\rangle . \tag{5.2}$$

Theorem 5.4. *Let us consider a system like* (2.1) *for which we know that it possesses m real roots with multiplicities $n_1, n_2, \ldots, n_m$, where $1 \leqslant n_j$ for $1 \leqslant j \leqslant m$. Then there exists a vector polynomial*

$$\begin{bmatrix} H_1(x) \\ \vdots \\ H_n(x) \end{bmatrix} \in (\mathbb{R}[x])^n \Big/ (Jf), \tag{5.3}$$

such that the system of equations

$$\begin{aligned} (f_1 + H_1)(x) &= 0, \\ &\vdots \\ (f_n + H_n)(x) &= 0, \end{aligned} \tag{5.4}$$

has $n_1 + n_2 + \cdots + n_m$ simple real roots.

Proof. We remark that

$$(f_i + H_i)(x) = \sum_{k=0}^{m_i} h_{i,k}(x')\, x_1^{m_i - k},$$

with $h_{i,m_i}(x') \not\equiv 0,\ h_{i,0}(x') \not\equiv 0,\ x' = (x_2, \ldots, x_n)$, after certain permutation of variables x. It is well known that there exists a perturbation $H_i(x)$ such that the equation

$$\sum_{k=0}^{m_i} h_{i,k}(x')\, x_1^{m_i - k} = 0$$

has m_i simple roots for a codimension 1 set of x' (cf. [2]). This fact entails that the system (5.4) also possesses as many simple roots as (2.1) has. □

Remark 5.5. One can understand this theorem in an intuitive way. Let us denote by $I_i := \operatorname{supp}(f_i + H_i)$ the set of powers present in the polynomial $f_i + H_i$. If the discriminant of the system

$$(f_i + H_i)(x) = \sum_{\alpha \in I_i} f_\alpha x^\alpha, \quad 1 \leqslant i \leqslant n, \tag{5.5}$$

say, $\Delta(f_\alpha) \in \mathbb{R}[f_{\alpha_1}, \ldots, f_{\alpha_{\sum_{i=1}^n |I_i|}}]$ does not vanish, then the roots of the system (5.5) are all simple. That is to say the set of the coefficients of the system (5.5) for which the system has multiple roots is of codimension one in the space of coefficients $\mathbb{R}^{\sum_{i=1}^n |I_i|}$. This fact is known under the name of Bertini–Sard theorem [2].

Remark 5.6. It is worth noting that one must choose a proper vector polynomial (5.3) to get distinct simple roots for the deformed system (5.4). For example if $f_1 = x^3$, the $f_1 = 0$ has a triple root at $x = 0$. If we take $H_1 = -x$, then $f_1 + H_1 = x^3 - x = x(x^2 - 1) = 0$ has three distinct roots $x = -1, 0, 1$, while for $H_1 = x$, the equation $x^3 + x = 0$ has one real simple root at $x = 0$ and two distinct complex roots.

The above proof on the existence of H_i is not constructive. As the example of the cubic equation shows, to specify the polynomials H_i is a difficult question on the real discriminant of the real polynomial system.

Example 5.7. Let us consider the system

$$\begin{cases} f_1(x_1, x_2) = x_1^2 - x_2^2 - 1 = 0, \\ f_2(x_1, x_2) = x_1^4 + x_2^2 - 1 = 0. \end{cases} \tag{5.6}$$

This system has two multiple real solutions $(\pm 1, 0)$ within the square $[-2, 2]^2$.

If we perturb this system with $H_1 = 0$ and the simple linear polynomial $H_2 = t(x_1 - 2)$, where $0 < t \leqslant 0.5$ as

$$\begin{cases} (f_1 + H_1)(x_1, x_2) = x_1^2 - x_2^2 - 1 = 0, \\ (f_2 + H_2)(x_1, x_2) = x_1^4 + x_2^2 + t(x_1 - 2) - 1 = 0, \end{cases} \tag{5.7}$$

then we have four simple real solutions. In particular, if we use the value $t = 0.5$ by applying the rootfinding method of [13, 14, 15], we obtain the following four real simple solutions:

$$\begin{aligned} &(\ \ 1.07123233675477, \pm 0.38410769233261), \\ &(-1.20970135357686, \pm 0.68071827127359). \end{aligned}$$

While if we use the value $t = 0.025$, we obtain the following four real simple solutions:

$$\begin{aligned} &(\ \ 1.00412951827050, \pm 0.09097301502177), \\ &(-1.01237171332486, \pm 0.15778620326351). \end{aligned}$$

Finally, if we use the value $t = 0.0125$, we obtain the following four real simple solutions:

$$\begin{aligned} &(\ \ 1.00207398824224, \pm 0.06443817123186), \\ &(-1.00621769007449, \pm 0.11168724107454). \end{aligned}$$

Example 5.8. Let us consider the system

$$
\left\{
\begin{aligned}
f_1(x_1,x_2,x_3) &= \big((x_1-x_2)^3 - x_3^2(x_1+x_2-x_3)\big)\\
&\quad \big((-x_1-x_2)^3 - x_3^2(-x_1+x_2-x_3)\big)\\
&\quad \big((-x_1+x_2)^3 - x_3^2(-x_1-x_2-x_3)\big)\\
&\quad \big((x_1+x_2)^3 - x_3^2(x_1-x_2-x_3)\big)\\
&= x_1^{12} - 6x_1^{10}x_2^2 + 15x_1^8x_2^4 - 20x_1^6x_2^6 + 15x_1^4x_2^8\\
&\quad - 6x_1^2x_2^{10} + x_2^{12} - 4x_1^{10}x_3^2 - 12x_1^8x_2^2x_3^2 + 56x_1^6x_2^4x_3^2\\
&\quad - 56x_1^4x_2^6x_3^2 + 12x_1^2x_2^8x_3^2 + 4x_2^{10}x_3^2 + 6x_1^8x_3^4\\
&\quad + 40x_1^6x_2^2x_3^4 + 164x_1^4x_2^4x_3^4 + 40x_1^2x_2^6x_3^4 + 6x_2^8x_3^4\\
&\quad - 6x_1^6x_3^6 - 50x_1^4x_2^2x_3^6 - 10x_1^2x_2^4x_3^6 + 2x_2^6x_3^6\\
&\quad + 5x_1^4x_3^8 - 2x_1^2x_2^2x_3^8 - 3x_2^4x_3^8 - 2x_1^2x_3^{10} - 2x_2^2x_3^{10}\\
&\quad + x_3^{12} = 0,\\
f_2(x_1,x_2,x_3) &= x_1^2 + x_2^2 - \frac{x_3^2}{2} = 0,\\
f_3(x_1,x_2,x_3) &= x_1^2 + \frac{x_2^2}{9} + \frac{x_3^2}{4} - 1 = 0.
\end{aligned}
\right. \tag{5.8}
$$

This system has eight simple real solutions

$$(\pm 0.25926718242254, \pm 1.21300057180546, \pm 1.75418919109753),$$

and eight triple real solutions

$$(\pm 0.68824720161168, \pm 0.68824720161168, \pm 1.37649440322337)$$

within the cube $[-2,2]^3$. We observe that these roots are invariant under the actions of a group $\Gamma := (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ generated by three generators $(x_1,x_2,x_3) \mapsto (-x_1,x_2,x_3)$, $(x_1,x_2,x_3) \mapsto (x_1,-x_2,x_3)$ and $(x_1,x_2,x_3) \mapsto (x_1,x_2,-x_3)$ due to the invariance of the system (5.8) itself under the same group action.

If we perturb this system with H_1 where $0 < t \leqslant 0.5$ and $H_2 = H_3 = 0$ as follows:

$$
\left\{
\begin{aligned}
&(f_1 + H_1)(x_1, x_2, x_3) \\
&\quad = \big((x_1 - x_2)^3 - t(x_1 - x_2) - x_3^2(x_1 + x_2 - x_3)\big) \\
&\qquad \big((-x_1 - x_2)^3 - t(-x_1 - x_2) - x_3^2(-x_1 + x_2 - x_3)\big) \\
&\qquad \big((-x_1 + x_2)^3 - t(-x_1 + x_2) - x_3^2(-x_1 - x_2 - x_3)\big) \\
&\qquad \big((x_1 + x_2)^3 - t(x_1 + x_2) - x_3^2(x_1 - x_2 - x_3)\big) \\
&\quad = t^4x_1^4 - 4t^3x_1^6 + 6t^2x_1^8 - 4tx_1^{10} + x_1^{12} - 2t^4x_1^2x_2^2 \\
&\qquad +4t^3x_1^4x_2^2 + 8t^2x_1^6x_2^2 + 12tx_1^8x_2^2 - 6x_1^{10}x_2^2 \\
&\qquad +t^4x_2^4 + 4t^3x_1^2x_2^4 + 4t^2x_1^4x_2^4 - 8tx_1^6x_2^4 \\
&\qquad +15x_1^8x_2^4 - 4t^3x_2^6 - 8t^2x_1^2x_2^6 - 8tx_1^4x_2^6 \\
&\qquad -20x_1^6x_2^6 + 6t^2x_2^8 + 12tx_1^2x_2^8 + 15x_1^4x_2^8 \\
&\qquad -4tx_2^{10} - 6x_1^2x_2^{10} + x_2^{12} + 4t^3x_1^4x_3^2 - 12t^2x_1^6x_3^2 \\
&\qquad +12tx_1^8x_3^2 - 4x_1^{10}x_3^2 - 28t^2x_1^4x_2^2x_3^2 \\
&\qquad +40tx_1^6x_2^2x_3^2 - 12x_1^8x_2^2x_3^2 - 4t^3x_2^4x_3^2 \\
&\qquad +28t^2x_1^2x_2^4x_3^2 + 56x_1^6x_2^4x_3^2 + 12t^2x_2^6x_3^2 \\
&\qquad -40tx_1^2x_2^6x_3^2 - 56x_1^4x_2^6x_3^2 \\
&\qquad -12tx_2^8x_3^2 + 12x_1^2x_2^8x_3^2 + 4x_2^{10}x_3^2 \\
&\qquad +6t^2x_1^4x_3^4 - 12tx_1^6x_3^4 + 6x_1^8x_3^4 \\
&\qquad +4t^2x_1^2x_2^2x_3^4 - 52tx_1^4x_2^2x_3^4 + 40x_1^6x_2^2x_3^4 \\
&\qquad +6t^2x_2^4x_3^4 - 52tx_1^2x_2^4x_3^4 + 164x_1^4x_2^4x_3^4 \\
&\qquad -12tx_2^6x_3^4 + 40x_1^2x_2^6x_3^4 + 6x_2^8x_3^4 - 2t^2x_1^2x_3^6 \\
&\qquad +8tx_1^4x_3^6 - 6x_1^6x_3^6 - 2t^2x_2^2x_3^6 + 24tx_1^2x_2^2x_3^6 \\
&\qquad -50x_1^4x_2^2x_3^6 - 10x_1^2x_2^4x_3^6 + 2x_2^6x_3^6 \\
&\qquad -4tx_1^2x_3^8 + 5x_1^4x_3^8 + 4tx_2^2x_3^8 \\
&\qquad -2x_1^2x_2^2x_3^8 - 3x_2^4x_3^8 - 2x_1^2x_3^{10} - 2x_2^2x_3^{10} + x_3^{12} = 0, \\
&(f_2 + H_2)(x_1, x_2, x_3) = x_1^2 + x_2^2 - \frac{x_3^2}{2} = 0, \\
&(f_3 + H_3)(x_1, x_2, x_3) = x_1^2 + \frac{x_2^2}{9} + \frac{x_3^2}{4} - 1 = 0,
\end{aligned}
\right.
\tag{5.9}
$$

then we have 32 simple real solutions. In particular, if we use the value $t = 0.5$, we obtain the following eight real solutions which are shifts of the simple solutions to system (5.8):

$$(\pm 0.27142016486929, \pm 1.20645760731621, \pm 1.74883324771051).$$

Also we obtain the following 24 simple real solutions

$$(\pm 0.68824720161168, \pm 0.68824720161168, \pm 1.37649440322337),$$
$$(\pm 0.78897550317143, \pm 0.32932116069209, \pm 1.20907797224513),$$
$$(\pm 0.44474589932680, \pm 1.07278013064881, \pm 1.64234961179579),$$

that originate from the triple solutions to system (5.8). We remark that the first ones of the above solutions coincide with the triple solutions to the original system. These roots are also invariant under actions of the group Γ. □

6. Concluding remarks

A problem concerning the perturbation of roots of a system of algebraic equations has been investigated. Its conservation and decomposition of a multiple root into simple roots have been discussed.

To this end, with our central Theorem 4.4 we show that the number of simple real roots of a system located in a compact set does not change after a sufficiently small perturbation of the system. This theorem can be applied to high dimensional CAD where it is sometimes needed to calculate intersection of several hypersurfaces that are perturbation of a set of original (unperturbed) hypersurfaces. For example, to draw a 3D (three dimensional) picture of a real algebraic surface that is obtained as a deformation of a known one, the question of the perturbation of roots plays an essential role [7, §9.6]. We hope that our results in this direction may be of interest to those who study the application of algebraic equations to computer graphics.

Furthermore, we give a result about the decomposition of multiple roots into simple roots. In particular, our Theorem 5.4 assures the existence of a deformed system (5.4) of the original system (2.1) that possesses only simple real roots. This result can be used in many cases including the computation of the topological degree [4, 6, 10, 11, 12] in order to examine the solution set of a system of equations and to obtain information on the existence of solutions, their number and their nature [1, 3, 6, 8, 9].

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