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On the equilibrium points of the relativistic restricted three–body problem

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Abstract

A detailed study of the relativistic problem of three bodies was presented by Brumberg ([1,2]). He derived the equations of motion for the general problem and he also deduced the corresponding equations for the restricted one. The existence and linear stability of triangular equilibrium points of the restricted problem were studied by Bhatnagar and Hallan ([3]). In this contribution we focus on the collinear libration points. We study the existence, position and stability of these points.

1 The equations of motion

In a synodic system of coordinates OXY, where O is the centre of mass of the two primaries, the motion of the test particle is described by the equations ([1,3])

$$\ddot{X} - 2n\dot{Y} = \frac{\partial\omega}{\partial X} - \frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{X}}\right), \quad \ddot{Y} + 2n\dot{X} = \frac{\partial\omega}{\partial Y} - \frac{d}{dt}\left(\frac{\partial\omega}{\partial\dot{Y}}\right), \tag{1}$$

where

$$\begin{aligned}\omega = &\frac{1}{2}n^2(X^2+Y^2) + \gamma\left(\frac{m_1}{R_1}+\frac{m_2}{R_2}\right)\\ &+\frac{1}{c^2}\left\{\frac{1}{8}[\dot{X}^2+\dot{Y}^2+2n(X\dot{Y}-Y\dot{X})+n^2(X^2+Y^2)]^2\right.\\ &+\frac{3\gamma}{2}\left(\frac{m_1}{R_1}+\frac{m_2}{R_2}\right)[\dot{X}^2+\dot{Y}^2+2n(X\dot{Y}-Y\dot{X})+n^2(X^2+Y^2)]\end{aligned}$$

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$$-\frac{\gamma^2}{2}\left(\frac{m_1^2}{R_1^2}+\frac{m_2^2}{R_2^2}\right)+\frac{\gamma m_1 m_2}{m_1+m_2}\left[na\left(4Y+\frac{7}{2}XY\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right.$$
$$+n^2a^2\left(-\frac{a}{R_1R_2}+\frac{m_2-2m_1}{2(m_1+m_2)R_1}+\frac{m_1-2m_2}{2(m_1+m_2)R_2}\right.$$
$$\left.\left.-\frac{Y^2m_2}{2(m_1+m_2)R_1^3}-\frac{Y^2m_1}{2(m_1+m_2)R_2^3}\right)\right]\Bigg\},$$

and

$\gamma =$ the constant of gravity,
m_1, m_2 are the masses of the two primaries,
$c =$ the velocity of light,
$a =$ the distance between the primaries,
$n =$ the mean motion $=$
$$= \frac{\gamma^{1/2}(m_1+m_2)}{a^{3/2}}\left[1-\frac{3\gamma(m_1+m_2)}{2c^2a}\left(1-\frac{m_1m_2}{3(m_1+m_2)^2}\right)\right],$$
$R_1 = \sqrt{(X-X_1)^2+Y^2}, \quad R_2 = \sqrt{(X-X_2)^2+Y^2},$
X_1, X_2 are the coordinates of the two primaries on the $OX-$axis.

We transform the above mentioned coordinate system to a dimensionless one, named Oxy, by choosing the unit of mass so that $m_1+m_2=1$ and the unit of time so that $\gamma=1$. The unit of distance is a ($a=1$). Then, if we denote by μ, $\mu \le 0.5$, the mass of the less massive primary, the mass of the other primary is equal to $1-\mu$ and their positions on the $Ox-$axis are $1-\mu$ and $-\mu$, respectively. It can be also derived that, in this coordinate system the "dimensionless" velocity of light and mean motion are

$$c_d = \frac{c}{\sqrt{\gamma(m_1+m_2)/a}}, \quad n_d = 1-\frac{3}{2c_d^2}\left(1-\frac{(1-\mu)\mu}{3}\right),$$

while the motion of the third particle is described by the system

$$\ddot{x}-2n_d\dot{y} =$$
$$x-\frac{(1-\mu)(x+\mu)}{r_1^3}-\frac{\mu(x+\mu-1)}{r_2^3}$$
$$+\frac{1}{c_d^2}\Bigg\{-3x+(1-\mu)\mu x-(\dot{x}-y)(x\dot{x}+y\dot{y}+x\ddot{y}-y\ddot{x}+\dot{x}\ddot{x}+\dot{y}\ddot{y})$$
$$+\frac{7(1-\mu)\mu}{2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)(x+2\dot{y}-\ddot{x})$$
$$-\frac{(1-\mu)\mu(-2+3\mu+8\dot{y}+7x)(x+\mu)}{2r_1^3}$$
$$-\frac{(1-\mu)\mu(1-3\mu-8\dot{y}-7x)(x+\mu-1)}{2r_2^3}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(x+2\dot{y}-\ddot{x}-\frac{3\,(1-\mu)\,(x+\mu)}{r_1^3}-\frac{3\mu\,(x+\mu-1)}{r_2^3}\right)\\
&\qquad\qquad\times\left(x^2+y^2+2\,(x\dot{y}-y\dot{x})+\dot{x}^2+\dot{y}^2\right)\\
&+3\,(-y+\dot{x})\left(\frac{(1-\mu)\,((x+\mu)\dot{x}+y\dot{y})}{r_1^3}+\frac{\mu\,((x+\mu-1)\dot{x}+y\dot{y})}{r_2^3}\right)\\
&+\frac{(1-\mu)\mu}{r_1r_2}\left(\frac{x+\mu}{r_1^2}+\frac{x+\mu-1}{r_2^2}\right)+\frac{(1-\mu)^2\,(x+\mu)}{r_1^4}+\frac{\mu^2\,(x+\mu-1)}{r_2^4}\\
&+\frac{3(1-\mu)\mu y^2}{2}\left(\frac{\mu(x+\mu)}{r_1^5}+\frac{(1-\mu)(x+\mu-1)}{r_2^5}\right)\Bigg\},
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\ddot{y}+2n_d\dot{x}=\\
&\quad y-\frac{(1-\mu)\,y}{r_1^3}-\frac{\mu y}{r_2^3}\\
&\quad+\frac{1}{c_d^2}\Bigg\{-3y+(1-\mu)\mu y-(x+\dot{y})\,(x\dot{x}+y\dot{y}+x\ddot{y}-y\ddot{x}+\dot{x}\ddot{x}+\dot{y}\ddot{y})\\
&\quad+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)(y-2\dot{x}-\ddot{y})\\
&\quad+\Big(-\mu y\,(-2+5\mu+7x+8\dot{y})+2\,((x+\mu)\,\dot{x}+y\dot{y})\,(4\mu+3\,(x+\dot{y}))\Big)\frac{1-\mu}{2r_1^3}\\
&\quad+\Big((1-\mu)y\,(-3+5\mu+7x+8\dot{y})\\
&\quad-2\,((x+\mu-1)\dot{x}+y\dot{y})\,(4(1-\mu)-3\,(x+\dot{y}))\Big)\frac{\mu}{2r_2^3}\\
&\quad+\frac{1}{2}\left(y-2\dot{x}-\ddot{y}-\frac{3(1-\mu)y}{r_1^3}-\frac{3\mu y}{r_2^3}\right)\left(x^2+y^2+2\,(x\dot{y}-y\dot{x})+\dot{x}^2+\dot{y}^2\right)\\
&\quad+\frac{(1-\mu)\mu y}{r_1r_2}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)+\frac{(1-\mu)^2y}{r_1^4}+\frac{\mu^2y}{r_2^4}\\
&\quad+\frac{3(1-\mu)\mu y^3}{2}\left(\frac{\mu}{r_1^5}+\frac{1-\mu}{r_2^5}\right)\Bigg\},
\end{aligned}
$$

where $r_1=\sqrt{(x+\mu)^2+y^2}$ and $r_2=\sqrt{(x+\mu-1)^2+y^2}$.

2 Position and stability of the collinear equilibrium points

The existence and position of the collinear equilibrium points of the problem are determined by the equation

$$
\begin{aligned}
&x-\frac{(1-\mu)(x+\mu)}{r_1^3}-\frac{\mu(x+\mu-1)}{r_2^3}\\
&+\frac{1}{c_d^2}\Bigg[-3x+(1-\mu)\mu x+\frac{x^3}{2}+\frac{7(1-\mu)\mu}{2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)x
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(1-\mu)(x+m)(m(-2+3\mu+7x)+3x^2)}{2r_1^3} \\
&-\frac{\mu(x+m-1)((1-m)(1-3\mu-7x)+3x^2)}{2r_2^3} \\
&+\frac{(1-\mu)\mu}{r_1 r_2}\left(\frac{x+\mu}{r_1^2}+\frac{x+\mu-1}{r_2^2}\right) \\
&+\left.\frac{(1-\mu)^2(x+\mu)}{r_1^4}+\frac{\mu^2(x+\mu-1)}{r_2^4}\right]=0,
\end{aligned} \tag{3}
$$

where $r_1 = |x+\mu|$ and $r_2 = |x+\mu-1|$. This equation results from System (2) if we put

$$
y = \dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0.
$$

In order to study the linear stability of the collinear equilibrium points, we transfer the origin of the coordinate system by putting

$$
x = x_L + \xi, \quad y = \eta, \tag{4}
$$

where x_L is the position of any collinear equilibrium point. Then, we linearize Equations (2) with respect to ξ, η and their derivatives. This linearization results in the following system

$$
\ddot{\xi} = Q_1\dot{\eta} + Q_2\xi, \quad \ddot{\eta} = Q_3\dot{\xi} + Q_4\eta, \tag{5}
$$

where

$$
\begin{aligned}
Q_1 &= \frac{A_2+2n_d}{1-A_1}, \quad Q_2 = \frac{A_3}{1-A_1}, \quad Q_3 = \frac{B_2-2n_d}{1-B_1}, \quad Q_4 = \frac{B_3}{1-B_1}, \\
A_1 &= -\frac{1}{2c_d^2}\left[x_L^2 + \frac{6(1-\mu)}{r_{1L}} + \frac{6\mu}{r_{2L}}\right], \\
A_2 &= \frac{1}{c_d^2}\left[2x_L^2 + \frac{6(1-\mu)}{r_{1L}} + \frac{6\mu}{r_{2L}} - \frac{3(1-\mu)x_L}{r_{1L}(x_L+\mu)}\right. \\
&\quad \left. -\frac{4(1-\mu)\mu(x_L+\mu)}{r_{1L}^3} + \frac{\mu(4(1-\mu)-3x_L)(x_L+\mu-1)}{r_{2L}^3}\right], \\
A_3 &= 1 + \frac{2(1-\mu)}{r_{1L}^3} + \frac{2\mu}{r_{2L}^3} + \frac{1}{c_d^2}\left[-3+\mu-\mu^2+\frac{3x_L^2}{2}+\frac{3(1-\mu)}{r_{1L}}+\frac{3\mu}{r_{2L}}\right. \\
&\quad -\frac{4(1-\mu)\mu+\mu(6x_L-7(1-\mu))(x_L+\mu)r_{1L}}{2r_{1L}r_{2L}(x_L+\mu)(x_L+\mu-1)} - \frac{(1-\mu)(12x_L+7\mu)}{2r_{1L}(x_L+\mu)} \\
&\quad +\frac{1-\mu}{2r_{1L}^3}(-\mu(4+\mu)+7\mu x_L+6x_L^2) - \frac{\mu(1-\mu)}{2r_{2L}^3}(5-\mu+x_L) \\
&\quad \left. -\frac{3(1-\mu)^2}{r_{1L}^4} - \frac{3\mu^2}{r_{2L}^4} - \frac{2(1-\mu)\mu}{r_{1L}r_{2L}}\left(\frac{1}{r_{1L}^2}+\frac{1}{r_{2L}^2}\right)\right],
\end{aligned}
$$

$$B_1 = -\frac{3}{2c_d^2}\left[x_L^2 + \frac{2(1-\mu)}{r_{1L}} + \frac{2\mu}{r_{2L}}\right],$$

$$B_2 = -\frac{1}{c_d^2}\left[2x_L^2 + \frac{6(1-\mu)}{r_{1L}} + \frac{6\mu}{r_{2L}} + \frac{4(1-\mu)\mu}{r_{2L}\,(x_L+\mu-1)} - \frac{(1-\mu)(x_L+\mu)(4\mu+3x_L)}{r_{1L}^3} - \frac{3\mu x_L(x_L+\mu-1)}{r_{2L}^3}\right],$$

$$B_3 = 1 - \frac{1-\mu}{r_{1L}^3} - \frac{\mu}{r_{2L}^3} + \frac{1}{c_d^2}\left[-3+\mu-\mu^2 + \frac{x_L^2}{2} + \frac{3(1-\mu)}{r_{1L}} + \frac{3\mu}{r_{2L}} - \frac{(1-\mu)(\mu(-2+5\mu)+7\mu x_L+3x_L^2)}{2r_{1L}^3} - \frac{\mu((1-\mu)(3-5\mu)-7(1-\mu)x_L+3x_L^2)}{2r_{2L}^3} + \frac{(1-\mu)^2}{r_{1L}^4} + \frac{\mu^2}{r_{2L}^4} + \frac{(1-\mu)\mu}{r_{1L}r_{2L}}\left(\frac{1}{r_{1L}^2}+\frac{1}{r_{2L}^2}\right)\right],$$

with $r_{1L} = |x_L+\mu|$ and $r_{2L} = |x_L+\mu-1|$.

We can easily transform Equations (5) to an equivalent system of order one. Then, the eigenvalues of the corresponding matrix of coefficients are given by

$$\lambda_{1,2} = \pm\sqrt{\frac{-R_1+\sqrt{R_1^2-4R_2}}{2}}, \quad \lambda_{3,4} = \pm\sqrt{\frac{-R_1-\sqrt{R_1^2-4R_2}}{2}}, \tag{6}$$

where

$$R_1 = -Q_1Q_3 - Q_4 - Q_2, \quad R_2 = Q_2Q_4.$$

These eigenvalues determine the kind of linear stability of the considered equilibrium points.

3 Numerical results

We have used the above mentioned analysis together with numerical methods to examine the existence, position and stability of the collinear equilibrium points for various cases of the model problem. It has been found that, in each case, there are three such points, named L_1, L_2 and L_3, whose positions fulfil the well–known relation $x_{L_3} < -\mu < x_{L_1} < 1-\mu < x_{L_2}$. In Table 1 we present these positions for all Sun–Planet pairs of our solar system. We also include the corresponding positions in the classical problem for comparison purposes (second entry in the table for each system). It can be seen that, in most of these cases, the positions of L_1 and L_2 are much more affected by the influence of the relativistic terms than that of the third equilibrium point. We have also computed the eigenvalues that determine the stability of these points. In all cases, two of these eigenvalues are real and opposite while the rest of them are imaginary. Consequently, the equilibrium points are unstable.

Table 1
Comparison of the positions of the collinear equilibrium points for several dynamical systems modeled by the use of the relativistic and the classical restricted problem.

μ	c_d	x_{L_1}	x_{L_2}	x_{L_3}
0.000000166000	6262.07	0.99619406057923	1.00381528794816	−1.00000006916666
(Sun–Mercury)		0.99619406054705	1.00381528798076	−1.00000006916666
0.000002447800	8560.03	0.99068234043513	1.00937097513753	−1.00000101991664
(Sun–Venus)		0.99068234039327	1.00937097518072	−1.00000101991666
0.000003003500	10064.84	0.99002657248316	1.01003413805726	−1.00000125145831
(Sun–Earth)		0.99002657245077	1.01003413809074	−1.00000125145833
0.000000322700	12424.24	0.99525140277101	1.00476303036243	−1.00000013445833
(Sun–Mars)		0.99525140276082	1.00476303037278	−1.00000013445833
0.000953692200	22947.35	0.93236993773108	1.06882613992582	−1.00039737170147
(Sun–Jupiter)		0.93236993769216	1.06882613997466	−1.00039737170283
0.000285726000	31050.90	0.95474919732924	1.04606932682937	−1.00011905249851
(Sun–Saturn)		0.95474919731454	1.04606932684648	−1.00011905249873
0.000043548000	44056.13	0.97576220622293	1.02454737493648	−1.00001814499997
(Sun–Uranus)		0.97576220621890	1.02454737494085	−1.00001814499999
0.000051668900	55148.85	0.97434749095228	1.02599374139635	−1.00002152870831
(Sun–Neptune)		0.97434749094956	1.02599374139930	−1.00002152870832
0.000000006500	63280.18	0.99870656252887	1.00129454074313	−1.00000000270833
(Sun–Pluto)		0.99870656252876	1.00129454074324	−1.00000000270833

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References

[1] V.A. Brumberg, Relativistic Celestial Mechanics, Press (Science), Moscow, Nauka, 1972.

[2] V.A. Brumberg, Essential Relativistic Celestial Mechanics, Adam Hilger, New York, 1991.

[3] K.B. Bhatnagar and P.P. Hallan, Celest. Mech. and Dynan. Astron. 69 (1998) 271.