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Symmetric doubly asymptotic orbits in the photogravitational restricted three-body problem

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Abstract

An interesting case of motion in dynamical systems is the case when moving bodies remain for an extremely long time in the vicinity of unstable equilibrium points of these systems by tracing orbits which asymptotically terminate on these equilibria. In this contribution we study this motion at the collinear equilibrium points of the photogravitational plane restricted three–body problem and give two types of doubly asymptotic orbits which have not been found to exist in the classical gravitational problem yet.

1 Introduction

In the classical circular plane restricted three–body problem it is known that the collinear equilibrium points L_1, L_2 and L_3 are always unstable. The asymptotic motion to these points was initially studied by Deprit and Henrard ([1]). They were concerned in asymptotic orbits "departing from" and "arriving at" these points and they indicated four possible types of "departure" or "arrival". Two of the combinations of these types determine orbits which are symmetric with respect to the axis containing the two main bodies while the rest give rise to non–symmetric orbits. Such orbits exist for certain values of the mass parameter of the problem and they are isolated. Both kinds of symmetric asymptotic motion are possible in the vicinity of L_3. In the cases of L_1 and L_2 only one of these two kinds has been found to exist, a different one in each case. Numerical results about these kinds of motion were presented in [1] and [2]. Here we consider the circular plane photogravitational three–body problem and, especially, the case when only the more massive main body emits radiation. In this problem we find that the particular kinds of asymptotic motion, which have not been found in the classical problem in the case of L_1 and L_2, may occur for certain combinations of the parameters of mass and radiation pressure.

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2 Equations of motion

In the usual barycentric, rotating and dimensionless coordinate system Oxy, with the two main bodies posed on the Ox–axis and having masses $1-\mu$ and μ ($\mu \leq 1/2$) and radiation pressure parameters q_1 and q_2 ($q_i \leq 1$, $i=1,2$), the motion of the third particle is described by the system (see [3], for example)

$$\ddot{x}-2\dot{y}=x(1-Q)-\mu(1-\mu)R, \quad \ddot{y}+2\dot{x}=y(1-Q), \tag{1}$$

where

$$Q=\frac{q_1(1-\mu)}{r_1^3}+\frac{q_2\mu}{r_2^3}, \quad R=\frac{q_1}{r_1^3}-\frac{q_2}{r_2^3}, \quad r_1=\sqrt{(x+\mu)^2+y^2}, \quad r_2=\sqrt{(x+\mu-1)^2+y^2}.$$

The collinear equilibrium points can be determined by solving the equations

$$x(1-Q)-\mu(1-\mu)R=0, \quad y=0. \tag{2}$$

We transfer the origin of the coordinate system to any collinear equilibrium point L by means of $x=x_L+\xi$, $y=\eta$. Then, Equations (1) are transformed to

$$\ddot{\xi}-2\dot{\eta}=(\xi+x_L)(1-Q)-\mu(1-\mu)R, \quad \ddot{\eta}+2\dot{\xi}=\eta(1-Q). \tag{3}$$

We expand the right parts of Equations (3) to Taylor series. Then, an approximation of these equations including all terms up to the fourth order is given by

$$\begin{aligned}
&\ddot{\xi}-2\dot{\eta}=A_1\xi+A_2\xi^2+A_3\eta^2+A_4\xi\eta^2+A_5\xi^3+A_6\xi^4+A_7\eta^4+A_8\xi^2\eta^2,\\
&\ddot{\eta}+2\dot{\xi}=B_1\eta+B_2\xi\eta+B_3\xi^2\eta+B_4\eta^3+B_5\xi\eta^3+B_6\xi^3\eta,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&A_1=1-Q_0-x_LQ_1-\mu(1-\mu)R_1, \quad A_2=-Q_1-4A_3, \quad A_3=x_LQ_2+\mu(1-\mu)R_2,\\
&A_4=Q_2-3(x_LQ_3+\mu(1-\mu)R_3), \quad A_5=4(x_LQ_3-Q_2+\mu(1-\mu)R_3),\\
&A_6=4(Q_3+2A_7), \quad A_7=-x_LQ_4-\mu(1-\mu)R_4, \quad A_8=-3(Q_3+4A_7),\\
&B_1=1-Q_0, \quad B_2=-Q_1, \quad B_3=-4Q_2, \quad B_4=Q_2, \quad B_5=-3Q_3, \quad B_6=4Q_3,\\
&Q_0=\frac{q_1(1-\mu)}{|a|^3}+\frac{q_2\mu}{|a-1|^3}, \qquad a=x_L+\mu,\\
&Q_1=-3\left(\frac{q_1(1-\mu)}{a|a|^3}+\frac{q_2\mu}{(a-1)|a-1|^3}\right), \quad Q_2=\frac{3}{2}\left(\frac{q_1(1-\mu)}{|a|^5}+\frac{q_2\mu}{|a-1|^5}\right),\\
&Q_3=\frac{5}{2}\left(\frac{q_1(1-\mu)}{a|a|^5}+\frac{q_2\mu}{(a-1)|a-1|^5}\right), \quad Q_4=\frac{15}{8}\left(\frac{q_1(1-\mu)}{|a|^7}+\frac{q_2\mu}{|a-1|^7}\right),\\
&R_1=-3\left(\frac{q_1}{a|a|^3}-\frac{q_2}{(a-1)|a-1|^3}\right), \quad R_2=\frac{3}{2}\left(\frac{q_1}{|a|^5}-\frac{q_2}{|a-1|^5}\right),\\
&R_3=\frac{5}{2}\left(\frac{q_1}{a|a|^5}-\frac{q_2}{(a-1)|a-1|^5}\right), \quad R_4=\frac{15}{8}\left(\frac{q_1}{|a|^7}-\frac{q_2}{|a-1|^7}\right).
\end{aligned}$$

We express the solution of System (4) in terms of a small orbital parameter ϵ

$$\xi(t)=\sum_i \xi_i(t)\epsilon^i, \quad \eta(t)=\sum_i \eta_i(t)\epsilon^i. \tag{5}$$

By substituting this solution in (4), we have that

$$\begin{aligned} &\ddot{\xi}_1-2\dot{\eta}_1=A_1\xi_1,\\ &\ddot{\eta}_1+2\dot{\xi}_1=B_1\eta_1, \end{aligned} \tag{6}$$

$$\begin{aligned} &\ddot{\xi}_2-2\dot{\eta}_2=A_1\xi_2+A_2\xi_1^2+A_3\eta_1^2,\\ &\ddot{\eta}_2+2\dot{\xi}_2=B_1\eta_2+B_2\xi_1\eta_1, \end{aligned} \tag{7}$$

$$\begin{aligned} &\ddot{\xi}_3-2\dot{\eta}_3=A_1\xi_3+2A_2\xi_1\xi_2+2A_3\eta_1\eta_2+A_4\xi_1\eta_1^2+A_5\xi_1^3,\\ &\ddot{\eta}_3+2\dot{\xi}_3=B_1\eta_3+B_2\xi_1\eta_2+B_2\xi_2\eta_1+B_3\xi_1^2\eta_1+B_4\eta_1^3, \end{aligned} \tag{8}$$

$$\begin{aligned} \ddot{\xi}_4-2\dot{\eta}_4=&\ A_1\xi_4+A_2\xi_2^2+2A_2\xi_1\xi_3+A_3\eta_2^2+2A_3\eta_1\eta_3+A_4\xi_2\eta_1^2+2A_4\xi_1\eta_1\eta_2+\\ &3A_5\xi_1^2\xi_2+A_6\xi_1^4+A_7\eta_1^4+A_8\xi_1^2\eta_1^2,\\ \ddot{\eta}_4+2\dot{\xi}_4=&\ B_1\eta_4+B_2\xi_2\eta_2+B_2\xi_1\eta_3+B_2\xi_3\eta_1+2B_3\xi_1\xi_2\eta_1+B_3\xi_1^2\eta_2+3B_4\eta_1^2\eta_2+\\ &B_5\xi_1\eta_1^3+B_6\xi_1^3\eta_1. \end{aligned} \tag{9}$$

The eigenvalues determining the linear stability of the collinear equilibria are

$$\lambda_{1,2,3,4}=\pm\frac{1}{2}\left(A_1+B_1-4\pm\sqrt{(A_1+B_1-4)^2-4A_1B_1}\right)^{1/2}.$$

Then, the general solutions of Systems (6-9) are expressed as follows

$$\xi_k(t)=\sum_{i=1}^{4} h_{ki}e^{k\lambda_i t}, \quad \eta_k(t)=\sum_{i=1}^{4} g_{ki}e^{k\lambda_i t}, \quad k=1,2,3,4. \tag{10}$$

By substituting these solutions in (5), we can approximate the motion in the vicinity of the collinear equilibrium points. For the determination of asymptotic orbits, at least one of the eigenvalues $\lambda_{1,2,3,4}$, say λ_1, must be positive. In this case, we use the solutions of Systems (6-9) along the eigenvector corresponding to λ_1

$$\xi_k(t)=h_{k1}e^{k\lambda_1 t}, \quad \eta_k(t)=g_{k1}e^{k\lambda_1 t}, \quad k=1,2,3,4. \tag{11}$$

We can easily find that

$$h_{11}=1, \qquad h_{21}=\frac{4B_2g_{11}\lambda_1+(4\lambda_1^2-B_1)\Phi_0}{16\lambda_1^4+4(4-A_1-B_1)\lambda_1^2+A_1B_1},$$

$$h_{31}=\frac{(9\lambda_1^2-B_1)\Phi_1+6\lambda_1\Phi_2}{81\lambda_1^4+9(4-A_1-B_1)\lambda_1^2+A_1B_1}, \qquad h_{41}=\frac{(16\lambda_1^2-B_1)\Phi_3+8\lambda_1\Phi_4}{256\lambda_1^4+16(4-A_1-B_1)\lambda_1^2+A_1B_1},$$

$$g_{11}=\frac{\lambda_1^2-A_1}{2\lambda_1}=\frac{2\lambda_1}{B_1-\lambda_1^2}, \qquad g_{21}=\frac{(4\lambda_1^2-A_1)B_2g_{11}-4\Phi_0\lambda_1}{16\lambda_1^4+4(4-A_1-B_1)\lambda_1^2+A_1B_1},$$

$$g_{31}=\frac{(9\lambda_1^2-A_1)\Phi_2-6\lambda_1\Phi_1}{81\lambda_1^4+9(4-A_1-B_1)\lambda_1^2+A_1B_1}, \qquad g_{41}=\frac{(16\lambda_1^2-A_1)\Phi_4-8\lambda_1\Phi_3}{256\lambda_1^4+16(4-A_1-B_1)\lambda_1^2+A_1B_1},$$

where

$$\begin{aligned}
&\Phi_0 = A_2 + A_3 g_{11}^2, \quad \Phi_1 = 2A_2 h_{21} + 2A_3 g_{11} g_{21} + A_4 g_{11}^2 + A_5,\\
&\Phi_2 = B_2 g_{21} + B_2 h_{21} g_{11} + B_3 g_{11} + B_4 g_{11}^3,\\
&\Phi_3 = A_2(h_{21}^2 + 2h_{31}) + A_3(g_{21}^2 + 2g_{11}g_{31}) + A_4(h_{21}g_{11}^2 + 2g_{11}g_{21}) + 3A_5 h_{21} + A_6 +\\
&\qquad A_7 g_{11}^4 + A_8 g_{11}^2,\\
&\Phi_4 = B_2(h_{21}g_{21} + h_{31}g_{11} + g_{31}) + B_3(g_{21} + 2h_{21}g_{11}) + 3B_4 g_{11}^2 g_{21} + B_5 g_{11}^3 + B_6 g_{11}.
\end{aligned}$$

Then, Solution (5) becomes

$$\xi(t) = \sum_{k=1}^{4} h_{k1} e^{k\lambda_1 t} \epsilon^k, \quad \eta(t) = \sum_{k=1}^{4} g_{k1} e^{k\lambda_1 t} \epsilon^k. \tag{12}$$

This solution approximates a doubly asymptotic orbit whose initial contitions are

$$\begin{aligned}
&\xi_0 = x_L + h_{11}\epsilon + h_{21}\epsilon^2 + h_{31}\epsilon^3 + h_{41}\epsilon^4, \qquad \eta_0 = g_{11}\epsilon + g_{21}\epsilon^2 + g_{31}\epsilon^3 + g_{41}\epsilon^4,\\
&\dot{\xi}_0 = (h_{11}\epsilon + 2h_{21}\epsilon^2 + 3h_{31}\epsilon^3 + 4h_{41}\epsilon^4)\lambda_1, \; \dot{\eta}_0 = (g_{11}\epsilon + 2g_{21}\epsilon^2 + 3g_{31}\epsilon^3 + 4g_{41}\epsilon^4)\lambda_1.
\end{aligned} \tag{13}$$

The kind of each asymptotic orbit depends on the sign of the parameter ϵ. This sign together with the linear aproximation of (12) ([1]) provide the possible types of the outgoing and incoming asymptotic orbits. In particular, symmetric doubly asymptotic orbits can be formed by

(a)	outgoing eigenvector	$\xi_1 = \epsilon e^{\lambda_1 t}$,	$\eta_1 = \epsilon d e^{\lambda_1 t}$,	and
	incoming eigenvector	$\xi_1 = \epsilon e^{-\lambda_1 t}$,	$\eta_1 = -\epsilon d e^{-\lambda_1 t}$,	or
(b)	outgoing eigenvector	$\xi_1 = -\epsilon e^{\lambda_1 t}$,	$\eta_1 = -\epsilon d e^{\lambda_1 t}$,	and
	incoming eigenvector	$\xi_1 = -\epsilon e^{-\lambda_1 t}$,	$\eta_1 = \epsilon d e^{-\lambda_1 t}$,	

where d depends on the coefficients of (6). In the classical problem and for L_3 both of these kinds are possible, in the case of L_1 only motion of kind (a) has been found, while for L_2 orbits of kind (b) are known. In the next section we will see that, in the photogravitational problem, both kinds are possible for L_1 and L_2, too.

3 Numerical investigation

We consider the case where $0 < q_1 < 1$, $q_2 = 1$. In this case it is found that there are three collinear equilibrium points, named L_1, L_2 and L_3, and their positions satisfy the relation $x_{L_3} < -\mu < x_{L_1} < 1-\mu < x_{L_2}$. These points are unstable ([3]).

Any symmetric doubly asymptotic orbit must fulfil the condition $\dot{x}(\mu, q_1) = 0$. To find an orbit of this kind, we set q_1 to a constant value, select a value for the mass parameter and, then, use (12) and (13) to get an initial approximation of the orbit. Then, by properly varying the value of μ, we can correct this approximation in order to find such an orbit for the specified value of q_1. We have calculated several doubly asymptotic orbits at L_1 and L_2 of both kinds for $q_1 = 0.8$. The elements of some of these orbits of the kind not found in the gravitational case, are given in Table 1. In this table we give the number of $Ox-$axis crossings of each orbit within its "half-period", the value of mass parameter for which this orbit exists, the position of the equilibrium point, the crossing point of the orbit at its "half–period" and the value of the Jacobi integral corresponding to this orbit.

Table 1

Symmetric doubly asymptotic orbits at L_1 ($\epsilon > 0,\ \dot{y}_0 < 0$), $q_1 = 0.8$

	N	μ	x_{L_1}	$x(T/2)$	C
1	2	0.00137395	0.88982188	2.81848828	−2.60991224
2	3	0.02030100	0.77268297	0.96338160	−2.76990274
3	4	0.01513875	0.79325918	2.71923220	−2.73654328
4	4	0.22801890	0.52889704	0.87187837	−3.14983285
5	4	0.08288543	0.61352326	1.26849946	−3.02951584
6	5	0.00124591	0.89201376	−2.56053312	−2.60799407
7	5	0.07878022	0.62189057	1.12803217	−3.01675566
8	6	0.00218867	0.87798048	5.73138654	−2.62123278
9	6	0.01770592	0.78263801	4.14167836	−2.75363041

Symmetric doubly asymptotic orbits at L_2 ($\epsilon < 0,\ \dot{y}_0 > 0$), $q_1 = 0.8$

	N	μ	x_{L_2}	$x(T/2)$	C
1	2	0.48240266	1.19580491	0.90626196	−3.34600690
2	5	0.38113414	1.22592548	1.21982592	−3.37471338
3	6	0.34245338	1.23570409	1.00975083	−3.37824906

Some of these orbits (labeled by their enumeration in the corresponding sub–table) are shown in Figure 1.

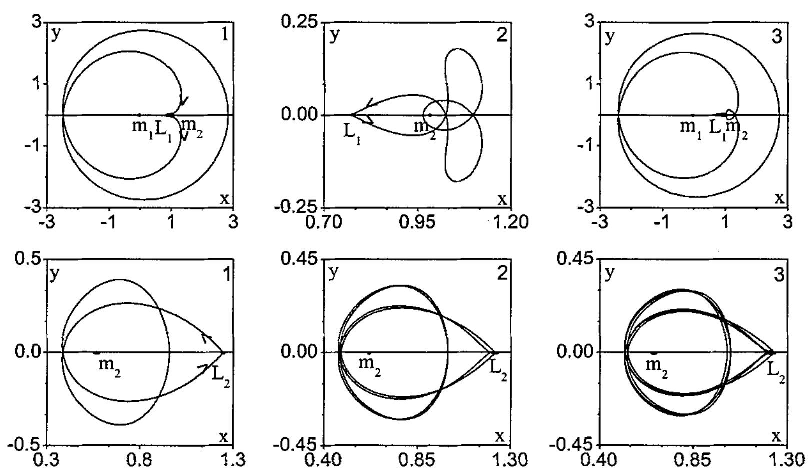

Fig. 1. Three orbits of type (b) departing from and ending at L_1, and three orbits of type (a) departing from and ending at L_2.

It is also interesting to investigate for which combinations of the mass and radiation parameters we should expect the existence of symmetric doubly asymptotic solutions departing from and approaching the collinear equilibrium points. This can be accomplished by calculating the relative variation of the mass and radiation pressure parameters along a series of orbits of this kind. We have calculated such a series in the case of L_1. This series contains orbits crossing twice the Ox−axis within their "half–period". The variation of q_1 with respect to μ is given in Figure 2.

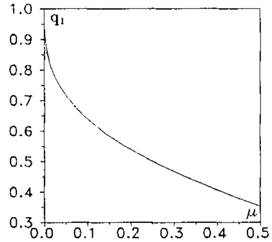

Fig. 2. The variation of q_1 versus μ along a series of symmetric doubly asymptotic orbits approaching L_1.

A number of these orbits are given in Table 2 while in Figure 3 we present some of them, labeled by their enumeration in the table. It can be seen that, as q_1 tends to unity, μ tends to zero and these orbits degenerate to a point. This explains the apparent lack of this kind of orbits in the classical restricted problem.

Table 2
Series with respect to q_1 of symmetric doubly asymptotic orbits at L_1

	μ	q_1	x_{L_1}	$x(T/2)$	C
1	0.00002056	0.98	0.97855724	1.01924801	−2.96235911
2	0.00242853	0.9	0.89360071	1.09542677	−2.84922231
3	0.01834207	0.8	0.78013212	1.18313192	−2.75770568
4	0.49999753	0.3527864	−0.11646847	1.36698603	−2.55554060

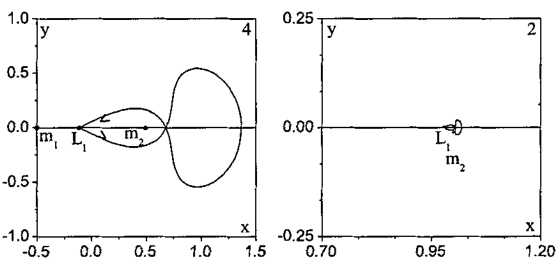

Fig. 3. Some elements of a series of symmetric doubly asymptotic orbits approaching L_1.

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