DEFLATION TECHNIQUES FOR THE DETERMINATION OF PERIODIC SOLUTIONS OF A CERTAIN PERIOD

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(Received 19 November 2002; accepted 19 February 2003)

Abstract. The computation of periodic orbits of nonlinear mappings or dynamical systems can be achieved by applying a root-finding method. To determine a periodic solution, an initial guess should be located within a proper area of the mapping or a surface of section of the phase space of the dynamical system. In the case of Newton or Newton-like methods these areas are the basins of convergence corresponding to the considered solution. When several solutions of the same period exist in a particular region, then the deflation technique is suitable for the calculation of all these solutions. This technique is applied here to the Hénon's mapping and the driven conservative Duffing's oscillator.

Keywords: nonlinear mappings, dynamical systems, periodic solutions, Poincaré maps, fixed points, root-finding methods, deflation techniques

1. Introduction

The numerical determination of periodic solutions in nonlinear mappings or dynamical systems is of great importance. Since these solutions are the zeros of a set of nonlinear equations, called the periodicity conditions, a suitable method to obtain them is to use a root-finding one. However, the basin of convergence of a root-finding method to a specific periodic solution is restricted to some small areas of the phase space due to the existence of several periodic solutions of the same or sub-multiple period close to each other. Thus, the determination of all periodic orbits of a certain period within a predefined region becomes a very difficult task, especially in the case when some of these orbits are unstable.

A suitable technique to face this problem is deflation. The general idea of this technique is the following: Once a solution of a nonlinear system has been determined, a modified system is formed in such a way that it retains all zeros of the original system except the zero that has been already computed. This procedure may be applied sequentially until all zeros of the original system are calculated.

In the special problem of finding periodic orbits of a dynamical system, deflation can be applied as follows: Suppose we have constructed a Poincaré surface of section of this system. On this surface, any periodic solution of period p is represented by p fixed points. All these points satisfy the system of periodicity

Astrophysics and Space Science **288:** 591–599, 2003.

conditions. Since stable solutions are expected to be more easily computed, we apply the root-finding method with an initial guess close to such a solution. Once this solution has been determined, the corresponding fixed points can be deflated from the original system of periodicity conditions in order to obtain a new system which will be fulfilled by the rest of the solutions.

The paper is organized as follows. In the next section we describe how root-finding methods and defiation techniques are applied for the determination of periodic solutions. In Section 3 we demonstrate the performance of such schemes for finding periodic orbits in the Hénon's mapping and Duffing's oscillator.

2. Root-finding Methods and Deflation Techniques

Periodic solutions of dynamical systems may be considered as sets of fixed points of properly constructed surfaces of section. Suppose that a surface of section is available for such a system and that Φ is the Poincaré map corresponding to this surface. We say that x, $x = (x_1, x_2, \ldots, x_n)$, is a fixed point of Φ, if for a certain $p \in \mathbb{Z}^+$

$$f(x) = x - \Phi^p(x) = 0 \tag{1}$$

Each periodic solution of period – p corresponds to p fixed points.

Suppose that an initial guess x^0 of one of these points is known. Then (1) can be solved by using a root-finding method to calculate the exact position of the fixed point, x_1. A suitable method for this calculation is Newton's method:

$$x^{k+1} = x^k - J_k^{-1} f(x^k), \quad k = 0, 1, 2, \ldots$$

where J_k denotes the Jacobian matrix of f evaluated at x^k. The elements of this matrix essentially depend on the partial derivatives of Φ^p with respect of x_i^k. These derivatives can be computed by means of the variational equations of the dynamical system, if they are available. If not, we may approximate them by finite difference schemes.

After calculating a specific fixed point of a periodic solution, the rest of them, x_j, $j = 2, 3, \ldots p$, can be easily obtained by successively applying the Poincaré map. Then, to determine a fixed point of another periodic solution of period p, we can eliminate the influence of the previously found orbit by deflating System (1). This can be accomplished by either using

$$\hat{f}_i(x) = \frac{f_i(x)}{\Pi_{j=1}^{p} \parallel x - x_j \parallel} = 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\parallel \cdot \parallel$ denotes the uniform or Euclidean norm, or

$$\hat{f}_i(x) = \frac{f_i(x)}{\Pi_{j=1}^{p} \langle \nabla f_i, (x - x_j) \rangle} = 0, \quad i = 1, 2, \ldots, n, \tag{3}$$

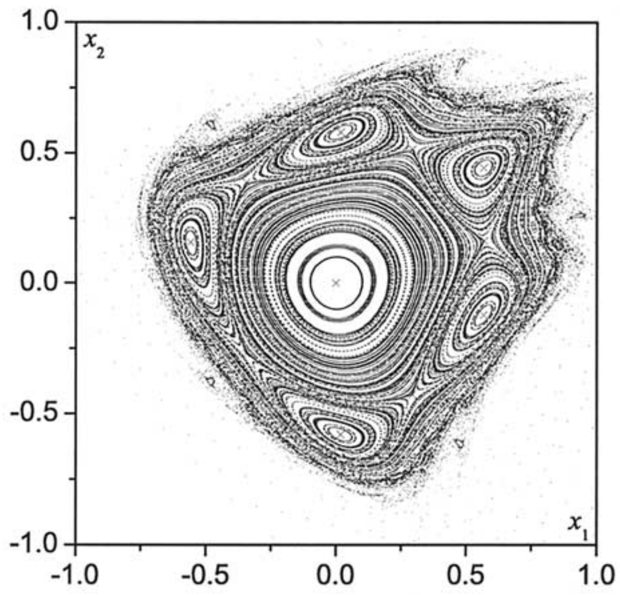

Figure 1. The Henón mapping for cos $a = 0.24$.

as a periodicity condition for the new solution. The deflation scheme given by (2) is usually called norm deflation while that expressed by (3) is called gradient deflation (Brown and Gearhart, 1971).

We should note that any orbit of a period which is sub-multiple of p should also be deflated since it acts upon this procedure as if it was of period p.

3. Applications

3.1. Hénon's mapping

The Hénon's quadratic area-preserving two-dimensional mapping is expressed as follows:

$$\Phi : \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - x_1^2 \end{pmatrix},$$

where $a \in [0, \pi]$ (Hénon, 1969). This mapping is presented in Figure 1 for cos $a = 0.24$. In this figure one can distinguish, among others, a stable periodic orbit of period 1 whose fixed point is located in the middle of the mapping. One may also perceive a stable periodic solution of period 5 whose fixed points lie in the centers of the five islands surrounding the 1-period orbit. An unstable periodic solution of period 5 also exists. This orbit is represented by the hyperbolic fixed points which are placed between the above mentioned islands. All these fixed points are given in Table I (Vrahatis, 1995).

TABLE I
Some fixed points of Hénon's mapping for $\cos a = 0.24$

N	Period	(x_1, x_2)		Stability
1	1	(0.00000000,	0.00000000)	S
2	5	(0.56724055,	–0.12232021)	S
		(0.56724055,	0.44408205)	
		(0.01739258,	0.58001859)	
		(–0.55859845,	0.15601611)	
		(0.01739258,	–0.57971609)	
3	5	(0.29421069,	–0.42748624)	U
		(0.56963265,	0.16224068)	
		(0.29421069,	0.51404617)	
		(–0.34438149,	0.38820846)	
		(–0.34438149,	–0.26960985)	

The fixed points together with the basins of convergence of Newton's method corresponding to them are presented in the first graph of Figure 2. The dark grey colored region represents the basin of the 5-period stable solution, the light grey colored region represents that of the 5-period unstable one, while the basin of convergence of the 1-period stable solution is the middle shade of gray one.

In the second graph of Figure 2 the basins of convergence corresponding to the two 5-period solutions are shown after deflating the 1-period one. It can be seen that a small vicinity around the deflated fixed point is now contained in the basin of convergence related to the fixed points of the stable 5-period orbit. Thus, the deflated point can be used as an initial guess for the calculation of this solution.

In the third graph of Figure 2 the basins of convergence of the unstable 5-period solution are shown after deflating the stable one. As in the previous case, all the fixed points of the deflated orbit are now included in these basins. So, any of these points is suitable as a starting point for the application of Newton's method.

3.2. DUFFING'S OSCILLATOR

The behavior of the driven conservative Duffing's oscillator is described by the equations (Drossos et al., 1996):

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3 + a \cos wt,$$

A Poincaré surface of section for this dynamical system is exhibited in Figure 3 for $a = 0.05$ and $w = 2.5$. In this surface several periodic solutions of different periods can be seen. The fixed points of six of them are given in Table II. The elements of stable orbits of period 1 (Table II entries numbered by 1 and 2), two

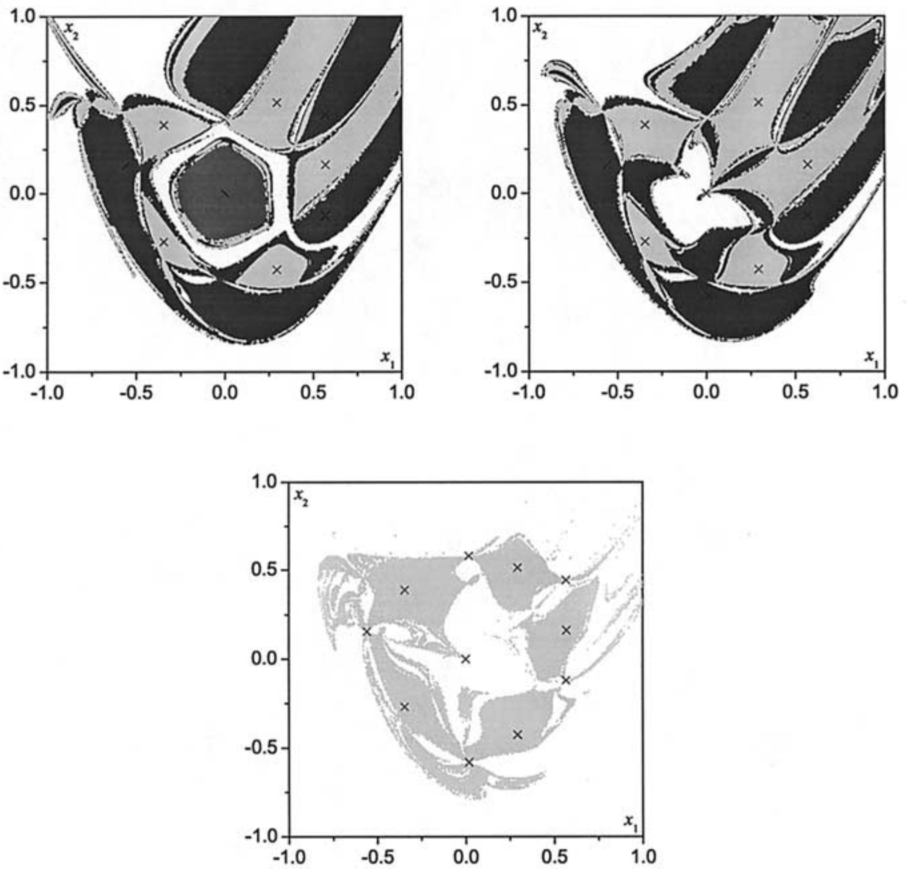

Figure 2. Application of deflation on Hénon's mapping for cos $a = 0.24$. In the first graph the basins of convergence of Newton's method for the fixed points of Table I are shown. These points are denoted by $\times$. In the second graph the 1-period solution is deflated. In the third graph the 5-period stable solution is deflated, too.

stable orbits of period 2 (entries numbered by 3 and 4) and two unstable orbits of period 2 (entries numbered by 4 and 5) are presented there.

The corresponding basins of convergence of Newton's method are given in Figure 4. The regions colored black denote the basins of the solutions of period 1 while the regions colored in different shades of grey represent the four orbits of period 2. Figure 5 shows the evolution of the basins of convergence which are related to the solutions of period 2 after deflating the two 1-period ones. It can be seen that the deflated fixed points belong now to the basins of convergence corresponding to the two solutions of period 2. The same behavior is observed if one deflates, in turn, the fixed points of the first and the second stable 2-period orbit (Figures 6 and 7) and, finally, the fixed points of one of the unstable solutions (Figure 8).

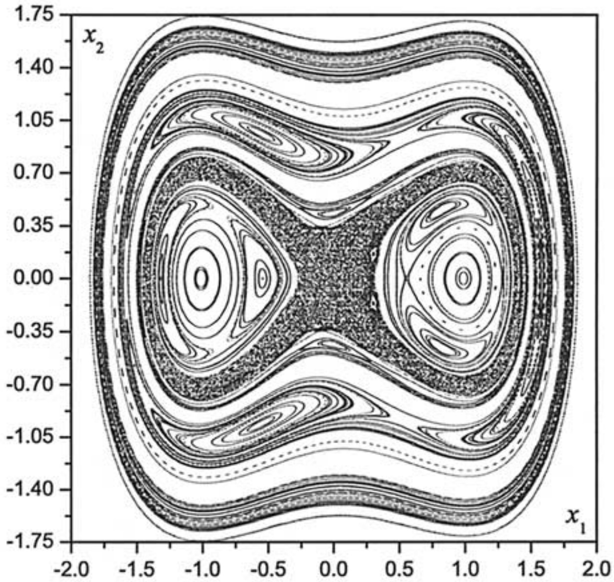

Figure 3. A Poincaré surface of section of Duffing's oscillator for $a = 0.05$ and $w = 2.5$.

TABLE II

Some fixed points of a Poincaré surface of section of Duffing's oscillator for $a = 0.05$ and $w = 2.5$

N	Period	(x_1, x_2)		Stability
1	1	(–1.01166859,	0.00000000)	*S*
2	1	(0.98814188,	0.00000000)	*S*
3	2	(–1.31896505,	0.00000000)	*S*
		(–0.54120403,	0.00000000)	
4	2	(0.82456378,	–0.45903976)	*S*
		(0.82456378,	0.45903976)	
5	2	(–0.86316920,	–0.44569502)	*U*
		(–0.86316920,	0.44569502)	
6	2	(0.55053928,	0.00000000)	*U*
		(1.28946962,	0.00000000)	

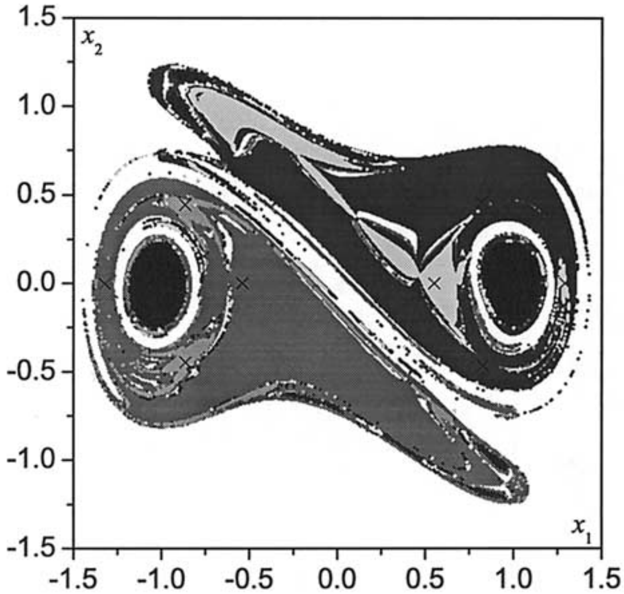

Figure 4. Basins of convergence of Newton's method for the fixed points given in Table II. The black colored basins correspond to the 1-period stable periodic solutions while the regions colored in different shades of grey represent the four orbits of period 2. The darker ones are related to the stable solutions. The exact location of the fixed points is marked by $\times$.

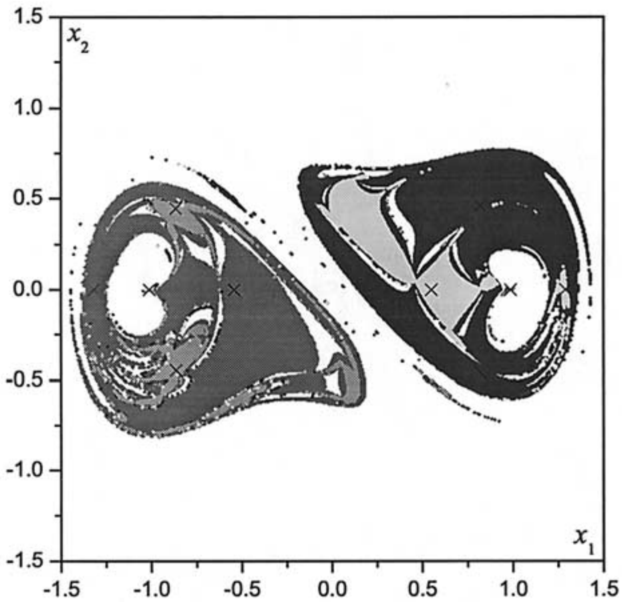

Figure 5. Basins of convergence of Newton's method for the fixed points given in Table II after deflating the two 1-period solutions.

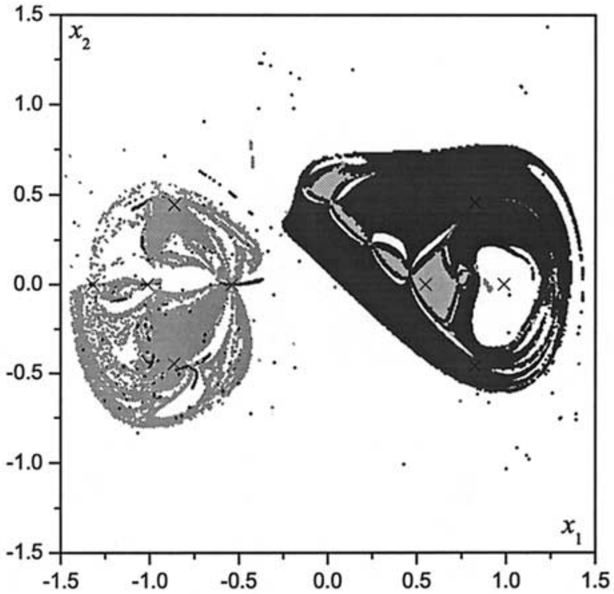

Figure 6. Basins of convergence of Newton's method after deflating the two 1-period solutions and one stable 2-period orbit.

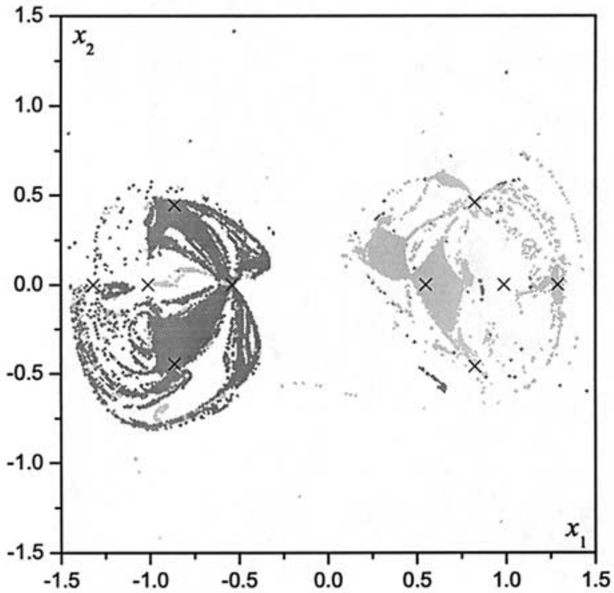

Figure 7. Basins of convergence of Newton's method for the unstable fixed points given in Table II after deflating all stable solutions.

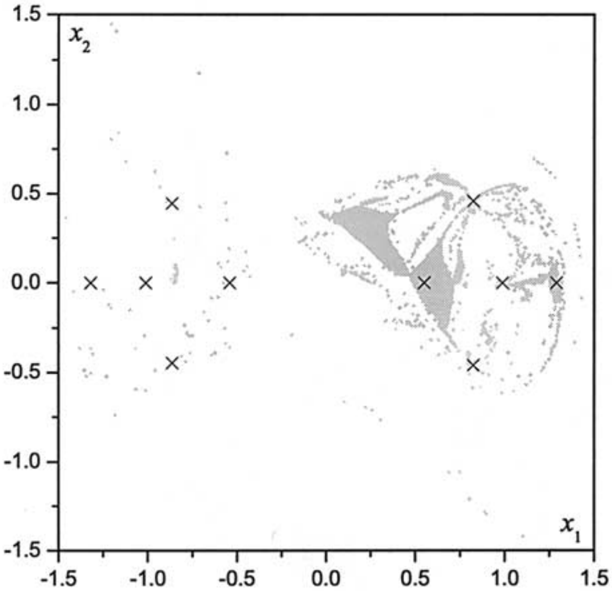

Figure 8. Basins of convergence of Newton's method for the fixed points of the last unstable solution given in Table II after deflating all the other solutions.

By following the same process we are able to compute any periodic solution of a certain period of Duffing's oscillator dynamical system.

Acknowledgements

One of the authors, V.S. Kalantonis, wishes to acknowledge the support granted to him by 'K. Karatheodory' research grants of the University of Patras.

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