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Non Commutative Quantum Mechanics in Time-Dependent Backgrounds

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1. Introduction

The problem of the dissipative systems from the quantum point of view has been of increasing interest, but it is far from having a satisfactory solution (1). There are many problems where the dissipation has an important role such as in quantum optics, in quantum analysis of fields, in quantum gravity (2). Dissipation can be observed in interactions between two systems, the observed system and another one often called reservoir or the bath, into which the energy flows via an irreversible manner (3). The system is embedded in some environment which is in principle unknown. For an effective description of such systems, we can use time dependent Hamiltonians which in classical physics yield the proper equation of motion. If the friction is a linear funcion of the velocity with friction constant γ the Hamiltonian is the well known Caldirola - Kanai Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m} e^{-\gamma t} \hat{p}^2 + V(\hat{q}) e^{\gamma t}$$

Some special potentials have been studied in reference (4). It should be noted however, that $\hat{\mathcal{H}}$ is not constant of motion and does not represent the energy of the system.
A second method for the dissipative systems, is based upon the procedure of doubling the phase space dimensions. The new degrees of freedom may assumed to represent the environment which absorb the energy dissipated by the dissipative system. H. Bateman (5) has shown that one can double the numbers of degrees of freedom so as to deal with an effective isolated system (6). In this article we assume that the coordinate operators q_1 and q_2, of these two systems respectively do not commute, that is $[\hat{q}_1, \hat{q}_2] = i\theta$ where θ is a real parameter and plays an analogous role of $\hbar$ in standard quantum mechanics.
For a manifold parametrized by the space - time coordinates x^μ, the commutation relations can be written as

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \qquad \mu, \nu = 0, ..., d$$

where $\theta^{\mu\nu}$ is a real antisymmetric tensor. This relation gives rise to the following space - time uncertainty relations

$$(\Delta x^\mu)\,(\Delta x^\nu) \geq \frac{1}{2}\,|\theta^{\mu\nu}|$$

The space - time non commutativity however violates causality, although could be consistent in string theory (9). Field theories with only space non commutativity, that is $\theta^{0i} = 0$ have a unitary S matrix, on the other hand theories with space - time non commutativity $\theta^{0i} \neq 0$ are not unitary. The non - commutativity of space, in the quantum field theory, appears through the modification of the product of the fields which appear in the action, by the so - called Moyal or star product (7),(8).

The idea of non commutative space - time was presented by Snyder (10) in 1947, with respect to the need to regularize the divergence of the quantum field theory. The idea was suggested by Heisenberg in 1930. It was Jon von Neumann , who began studying this "pointless geometry". The physical theories of today they hold only in empty space which in reality does not exist. The seed of the original idea goes back to ancient Greek Stoik Philosophers, especially to Zeno the Kitieus, who contrary to the followers of Democritus, said that there is not empty space (11).

In the past few years there has been an increasing interest in the non commutative geometry, extensively developed by Connes (12), for the study of many physical problems. It has become clear that there is a strong connection of these ideas with string theories (13), finding many applications in solid state and particles physics. The non commutative geometry arises very naturally from the Matrix theory (14).

There is another immediate motivation of noncommutative theories in quantum gravity. Classical general relativity breaks down at Planck scale l_p, where quantum effects become important.

$$l_p = \sqrt{\frac{hG}{c^3}} \simeq 10^{-33} cm \qquad t_p = \frac{l_p}{c} \simeq 10^{-44} sec$$

Einstein's theory implies that gravity is equivalent to spacetime geometry. Hence quantum gravity should quantize spacetime and spacetime quantization requires to promote coordinates to hermitian operators which do not commute. The wave function actually becomes an operator. The point is replaced by some "cell" and thus the spacetime becomes fuzzy at very short distances. It is apparent that this conflicts with Lorentz invariance. Physics near Planck dimensions is not yet fully understood. At these dimensions the cone of light acts as if it were fuzzy and we cannot distinguish between the past and the future. A review of recent efforts to add a gravitational field to non commutative models can be found in (15) and references therein.

Non commutativity is the central mathematical concept expressing the uncertainty. The phase space of ordinary quantum mechanics is a well known example of non commuting space. The momenta of a system in the presence of a magnetic field act as non commuting operators as well. The canonical commutation relations of quantum mechanics introduce a cellular structure in phase space. Noncommuting coordinates will introduce a cellular structure in ordinary space as well, similar to a lattice structure, which a priory lead to a nonlocal theory. This nonlocality however may be desirable since there are reasons to believe that any theory of quantum gravity will not be local in the conventional sense. Anyway it is a long - held belief that in quantum theories, space - time must change its nature at distances compared to the Planck scales.

The experimental signature of noncommuting spatial coordinates which is currently available seems to be the approximate noncommutativity appearing in the Landau problem, the lowest energy levels, for the case of very strong magnetic field. The study of exactly solvable models, as is for instance the present note, should lead to a better understanding of some issues in noncommutative theories.

In recent years there has been increasing interest in quantizing the harmonic oscillator with a variable mass in a time varying crossed electromagnetic field (16). In this article we will study the problem of a two dimensional time dependent harmonic oscillator within non

commutative quantum mechanics (17) (18), the parameters of which are also time dependent. All the time dependent factors are exponentials of the form $e^{\gamma t}$. These factors have been chosen so that the resulting final formulas, fluctuate with some external frequencies which do not depend on time.
We postulate first the two dimensional phase space. The momenta commutator shows that we have a time dependent magnetic field. With a time dependent, Bopp shift, linear transformation we reduce the phase space to a new phase space with two independent dimensions. The coordinates and the momentum of the second dimension of this new phase space satisfy a deformed time independent commutation relation. Next we give the Hamiltonian of the system which is a two dimensional damped harmonic oscillator. It is actually a linear combination of two Caldirola - Kanai damped harmonic oscillators with friction terms. Following that, we find the exact propagator of the system and the time evolution of the basic operators. In the next section we find the propagator in the case where the deformed parameter μ vanishes and the system becomes one dimensional. Finally in the last section we find the statistical partition function for two distinct particular cases which differ in the number of the dimensions but the final results depend on one common frequency.

2. The two dimensional phase space

We postulate the following time dependent commutation relations.

$$[\hat{p}_1, \hat{p}_2] = i\lambda e^{-\gamma_1 t}, \qquad [\hat{q}_1, \hat{q}_2] = i\theta e^{\gamma_1 t}, \qquad [\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk} \tag{1}$$

Recently non commutative theories with non constant parameters have been found in some references such as (19). The commutators of the pairs q_1, p_2 and q_2, p_1 are zero and the system can be described initially by one of the following wave functions $\psi(q_1, p_2, 0)$ or $\psi(q_2, p_1, 0)$.
The first relation among momenta operators means that the system is in a time dependent magnetic field, ($\lambda \sim B$), perpendicular to the plane of q_1 and q_2. The magnetic field is defined in terms of the symmetric time dependent potential $\vec{A} = e^{-\gamma_1 t}(-Bq_2/2,\ Bq_1/2,\ 0)$, which indicates that there is also an electric field present.

$$\vec{B} = \vec{\nabla} \times \vec{A} = e^{-\gamma_1 t} B\vec{k} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \gamma_1 \vec{A}$$

The second commutation relation between the coordinates, expresses the non commutativity of space. The parameter θ has a dimension of (length)^2. In lowest Landau levels the coordinates of the plane are also canonical conjugate operators and so satisfies the same relation with $\theta \sim (1/B)$ (20). This commutation relation implies the following uncertainty.

$$\Delta q_1 \Delta q_2 \sim (\theta/2) e^{\gamma_1 t} \tag{2}$$

So the position of the system cannot be localized in space, except for minus infinite times. The coordinates space becomes now fuzzy and fluid. The parameter θ represents the fuzziness and the parameter γ_1 the fluidity of the space.
The above relation is the known relation of ordinary non commutative geometry except that the parameter $\theta \sim \theta(t)$ expands exponentially with the evolution of time. This is the main motivation of this paper. The effect of a changing magnetic field is given by Faraday's law which states that the magnetic flux running through a closed loop may change because the field itself changes or because the loop is moving in space. So if we focus on point one, that is

$\Delta q_1 \to \epsilon > 0$ then the second point becomes fuzzier as time passes, that is $\Delta q_2 \to \infty$. We can say that the background space is a dynamical two - dimensional fuzzy space (21).
We will transform the problem of the non commuting two dimensional space, to a problem of two coupled harmonic oscillators in a more familiar two dimensional quantum mechanical space. The two dimensions of the new phase space are now independent of each other, but the second phase space satisfies a deformed commutation relation.
For this purpose we make the following linear transformations

$$\hat{P}_1 = \hat{p}_1 \qquad \hat{Q}_1 = \hat{q}_1$$
$$\hat{P}_2 = \hat{p}_2 + e^{-\gamma_1 t}\frac{\lambda}{\hbar}\hat{q}_1 \qquad \hat{Q}_2 = \hat{q}_2 - e^{\gamma_1 t}\frac{\theta}{\hbar}\hat{p}_1 \tag{3}$$

The commutators for the basic operators within the capital letters are

$$[\hat{Q}_1, \hat{Q}_2] = 0 \qquad [\hat{P}_1, \hat{P}_2] = 0$$
$$[\hat{Q}_1, \hat{P}_1] = i\hbar \qquad [\hat{Q}_2, \hat{P}_2] = i\left(\hbar - \frac{\lambda\theta}{\hbar}\right) = i\hbar\left(1 - \frac{\lambda\theta}{\hbar^2}\right) = i\hbar\mu \tag{4}$$

The basic operators of the second dimension $\hat{P}_2$, $\hat{Q}_2$ satisfy a deformed commutation relation which it is time independent. We denote the deformed parameter by μ. We emphasize that the time dependence of the commutation relation in equations (1), are chosen so that this new parameter $\mu = 1 - (\lambda\theta/\hbar^2)$ becomes time independent. If this parameter becomes zero, $\mu = 0$ that is $\lambda\theta = \hbar^2$, the four - dimensional phase space degenerates to a two - dimensional one and consequently as we will see later, the final solutions depend only on one frequency Ω. The canonical limit $(\theta \to 0, \lambda \to 0)$ does not exist in this case. It is clear that we have to keep both parameters non zero, $\lambda \neq 0$ and $\theta \neq 0$.
Instead of an algebra of commutators, some theoretical physicists (22),(23) consider its classical analogon involving Poisson brackets of functions on real variables. But the standard limit from quantum to classical mechanics that is $\hbar \to 0$ has as a result to vanish the third of the commutators (4), while the last one tends to infinity. So the noncommutative quantum mechanics seems to has no classical analogous or we have to treat the second phase spaces with some deferent manner as an extra dimension. It seems reasonable to make the substitutions $\theta \to \theta\hbar$ and $\lambda \to \lambda\hbar$ so that the limit $\hbar \to 0$ vanish all the commutators collectively.
We see that the time dependence on the magnetic field makes the time dependence of the coordinates commutator unavoidable, rendering the resulting commutator of the capital operators $\hat{P}_2$ and $\hat{Q}_2$ time independent. The system is now described initially on the coordinates space or on the momentum space by one of the following wave functions

$$\Psi_0(Q_1, Q_2) = \Psi_0(q_1, q_2 - \frac{\theta}{\hbar}e^{\gamma_1 t}p_1),$$
$$\Psi_0(P_1, P_2) = \Psi_0(p_1, p_2 + \frac{\lambda}{\hbar}e^{-\gamma_1 t}q_1) \tag{5}$$

The time dependence of the initial wave functions is due to the moving phase space of the second point (Q_2, P_2), resulting from the time dependent magnetic field.

3. The damped Hamiltonian of the system

We will use the following Hamiltonian

$$\hat{\mathcal{H}}(\hat{\vec{p}},\hat{\vec{q}},t) = e^{2(\gamma_1+\gamma_2)\,t}\,\frac{\hat{p}_1^2}{2m_1} + e^{-2(\gamma_1+\gamma_2)\,t}\,\frac{1}{2}m_1\omega_1^2\hat{q}_1^2 - \kappa\left(e^{-2\gamma_2\,t}\,\frac{\hat{p}_2^2}{2m_2} + e^{2\gamma_2\,t}\frac{1}{2}m_2\omega_2^2\hat{q}_2^2\right) \tag{6}$$

which is usually referred to as the Caldirola - Kanai model (24). The coupling constant κ will take one of the values ± 1. For $\kappa = -1$ the Hamiltonian is an ordinary two dimensional Hamiltonian. For $\kappa = 1$ we can say that the second Hamiltonian is an harmonic oscillator with negative mass $m_2 < 0$ (25).

The damped harmonic oscillator in a crossed magnetic field in ordinary space has been studied by many authors (26). We will study this problem in non commutative quantum mechanics. Such problems with magnetic fields in noncommutative quantum mechanics has also been studied by some authors see for instance (27), (28), and references there in.

We can assume that this is a Hamiltonian of two particles, one on the phase point (q_1, p_1) and the other on the point (q_2, p_2). It has been shown, by Bateman, that we can apply the usual canonical quantization method if we double the numbers of degrees of freedom so as to deal with an effective isolated system. The new degrees of freedom may be assumed to represent the environment which absorbs the energy dissipated by the dissipative system and the time dependent magnetic fields. The canonical quantization of these dual Beteman's type systems have many problems which have been pointed out in the relevant literature (29). We think that the non vanishing commutator of the coordinate operators q_1 and q_2, corrects many of these problems. Notice that as it was pointed out earlier, the Caldirola - Kanai Hamiltonian is not really the energy of the system but rather an operator which generates the motion of the system.

For the case where $\gamma_1 = 0$ the Hamiltonian becomes symmetric or antisymmetric with the reversal of time. This case has been studied in ref (31). The presence of the γ_1 parameter obviously breaks down this very important symmetry and indicates the presence of an electric field.

The Hamiltonian (6) describes a system of two particles with varying masses of $m_1 \to m_1 e^{-2(\gamma_1+\gamma_2)\,t}$ and $m_2 \to m_2 e^{2\gamma_2}$ respectivelly. The product of these two masses is obviously time dependent

$$m_1 m_2 \to m_1 m_2\, e^{-2\gamma_1\,t} \tag{7}$$

This is the consequence of the time varying magnetic field $B(t) \to Be^{-\gamma_1\,t}$. The time factors of this Hamiltonian have been chosen so that the finally oscillators of the system become time independent.

With the linear transformation (3) the Hamiltonian becomes

$$\hat{\mathcal{H}}(\hat{\vec{P}},\hat{\vec{Q}},\,t) = \frac{1}{2}\,e^{2(\gamma_1+\gamma_2)\,t}\hat{P}_1^2(1-\kappa\omega_2^2\theta^2) + \frac{1}{2}\,e^{-2(\gamma_1+\gamma_2)\,t}\hat{Q}_1^2(\omega_1^2-\kappa\lambda^2) - \frac{1}{2}\,e^{-2\gamma_2 t}\kappa\hat{P}_2^2 - \frac{1}{2}\,e^{2\gamma_2 t}\kappa\omega_2^2\hat{Q}_2^2 - e^{(\gamma_1+2\gamma_2)\,t}\kappa\omega_2^2\theta\hat{P}_1\hat{Q}_2 + e^{-(\gamma_1+2\gamma_2)\,t}\kappa\lambda\hat{P}_2\hat{Q}_1 \tag{8}$$

where we have set $\hbar = 1$ and $m_1 = m_2 = 1$.

This is a Hamiltonian of two coupled harmonic oscillators in the deformed quantum mechanical space. The last two factors of the above Hamiltonian are the coupling terms. Notice that if $\gamma_1 + 2\gamma_2 = 0$ these terms become time independent.
In order to simplify the relations we shall use the following symbolism

$$\hat{P}_1 \to \hat{T}_1 \qquad \hat{Q}_1 \to \hat{T}_2 \qquad \hat{P}_2 \to \hat{T}_3 \qquad \hat{Q}_2 \to \hat{T}_4 \tag{9}$$

The commutation relations become

$$[\hat{T}_2, \hat{T}_1] = c_{21} \qquad [\hat{T}_4, \hat{T}_3] = c_{43}, \qquad c_{21} = i\hbar \qquad c_{43} = c_{21}\mu \tag{10}$$

while all the others commutators vanish. The last commutation relation goes to infinity as $\hbar \to 0$ so the problem has no classical analogy (Figure 1).

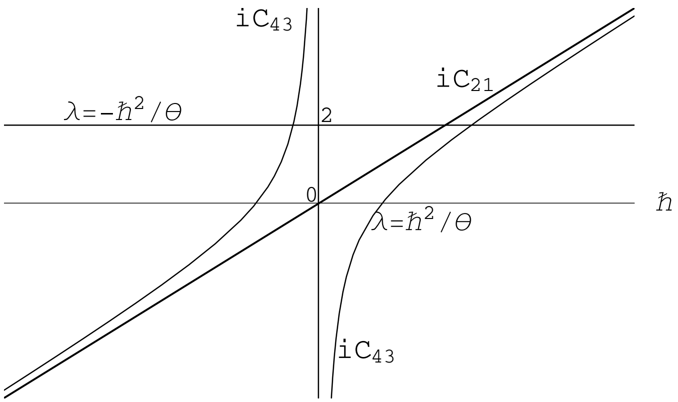

Fig. 1. The commutators $i\,c_{21}$ and $i\,c_{43}$ of the phase space as functions of $\hbar$.

The Hamiltonian is written

$$\hat{\mathcal{H}}(\hat{\vec{T}}) = k_{11}\hat{T}_1^2 + k_{22}\hat{T}_2^2 + k_{33}\hat{T}_3^2 + k_{44}\hat{T}_4^2 + k_{41}\hat{T}_4\hat{T}_1 + k_{32}\hat{T}_3\hat{T}_2 \tag{11}$$

Where we have set

$$k_{11} = \frac{1}{2}\, e^{2(\gamma_1+\gamma_2)\,t}(1 - \kappa\omega_2^2\theta^2), \qquad k_{22} = \tfrac{1}{2}\, e^{-2(\gamma_1+\gamma_2)\,t}(\omega_1^2 - \kappa\lambda^2)$$

$$k_{33} = -\frac{1}{2}\, e^{-2\gamma_2 t}, \qquad k_{44} = -\tfrac{1}{2}\, e^{2\gamma_2 t}\kappa\omega_2^2$$

$$k_{32} = e^{-(\gamma_1+2\gamma_2)\,t}\kappa\lambda, \qquad k_{41} = -e^{(\gamma_1+2\gamma_2)\,t}\kappa\omega_2^2\theta \tag{12}$$

The coupling terms k_{32} and k_{41} become zero in the case of commutative quantum mechanics, where $\lambda \to 0$ and $\theta \to 0$.

4. The exact propagator of the system

We will expand the time evolution operator in an appropriate ordered form so that the propagator will be calculated easily with a straight manner.
The Shrödinger equation of motion of the time evolution operator is:

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t)=\hat{\mathcal{H}}(t)\hat{U}(t)\qquad\qquad \hat{U}(0)=1$$

We look for the following, normal ordered expansion of the evolution operator:

$$\hat{U}(t)=\hat{U}_4\hat{U}_3\hat{U}_2\hat{U}_1$$

where

$$\hat{U}_4=e^{f_{44}\hat{T}_4^2}e^{\frac{1}{2}f_{43}(\hat{T}_4\hat{T}_3+\hat{T}_3\hat{T}_4)}e^{f_{42}\hat{T}_4\hat{T}_2}e^{f_{41}\hat{T}_4\hat{T}_1}\qquad \hat{U}_3=e^{f_{33}\hat{T}_3^2}e^{f_{32}\hat{T}_3\hat{T}_2}e^{f_{31}\hat{T}_3\hat{T}_1}$$
$$\hat{U}_2=e^{f_{22}\hat{T}_2^2}e^{\frac{1}{2}f_{21}(\hat{T}_2\hat{T}_1+\hat{T}_1\hat{T}_2)}\qquad \hat{U}_1=e^{f_{11}\hat{T}_1^2}\tag{13}$$

The functions $f_{jk}(t)$ are time dependent and because of the condition $\hat{U}(0)=1$, they satisfy the initial conditions $f_{jk}(0)=0$.
The above solution is always possible if the Hamiltonian takes the following form $\hat{\mathcal{H}}=\sum_{i=1}^{m}a(t)\hat{H}_i$ where the operators $\hat{H}_i$, $i=1,...,m$ forms a closed Lie algebra (30). We have proved in reference (32) using standard algebraic technics that all the unknown functions f_{jk} can be written with the help of the functions $x_{jk}(t)$. The functions $x_{jk}(t)$ satisfy the following classical differential system

$$\begin{aligned}
x'_{1j}&=-2k_{11}x_{2j}+\mu\,k_{41}x_{3j}\\
x'_{2j}&=2k_{22}x_{1j}-\mu\,k_{32}x_{4j}\\
x'_{3j}&=k_{32}x_{1j}-2\mu\,k_{33}x_{4j}\\
x'_{4j}&=-k_{41}x_{2j}+2\mu\,k_{44}x_{3j}
\end{aligned}\tag{14}$$

with the following initial conditions

$$x_{jk}(0)=\delta_{jk}\tag{15}$$

The functions x_{nj} give the time evolution of the basic operators $\hat{T}_j$. We have

$$\hat{T}_n(t)=e^{-\frac{i}{\hbar}t\hat{\mathcal{H}}}\hat{T}_j(0)e^{\frac{i}{\hbar}t\hat{\mathcal{H}}}=x_{nj}(t)\hat{T}_j(0)\tag{16}$$

The solution of the above system is as follows

$$\begin{aligned}
x_{11}&=e^{(\gamma_1+\gamma_2)\,t}\left[a_2-(\gamma_1+\gamma_2)b_2-\kappa^2\omega_2^2\mu\left(a_1-(\gamma_1\mu+\gamma_2)b_1\right)\right]\\
x_{22}&=e^{-(\gamma_1+\gamma_2)\,t}\left[a_2+(\gamma_1+\gamma_2)b_2-\kappa^2\omega_2^2\mu\left(a_1+(\gamma_1\mu+\gamma_2)b_1\right)\right]\\
x_{21}&=e^{(\gamma_1+\gamma_2)\,t}\left[\kappa^2\omega_2^2\mu^2b_1-b_2+\kappa\omega_2^2\theta^2b_2\right]\\
x_{12}&=e^{-(\gamma_1+\gamma_2)\,t}\left[-\omega_1^2(\kappa^2\omega_2^2\mu^2b_1-b_2)-\kappa\lambda^2b_2\right]
\end{aligned}$$

$$
\begin{aligned}
x_{13} &= e^{-\gamma_2 t}\left[\omega_2^2\left(\kappa^2\lambda\mu^2 b_1 - \kappa\theta\mu(\gamma_1(a_1+\gamma_2 b_1)+b_2)\right)\right] \\
x_{14} &= e^{\gamma_2 t}\left[-\kappa\mu\lambda(a_1-\gamma_2 b_1)+\kappa^2\omega_2^2\theta\mu\left(a_1-(\gamma_1\mu+\gamma_2)b_1\right)\right] \\
x_{23} &= e^{-\gamma_2 t}\left[\omega_2^2\left(\kappa\omega_1^2\theta\mu(a_1+\gamma_2 b_1)-\kappa^2\lambda\mu(a_1+(\gamma_1\mu+\gamma_2)b_1)\right)\right] \\
x_{24} &= e^{\gamma_2 t}\left[\kappa^2\omega_1^2\omega_2^2\theta\mu^2 b_1+\kappa\lambda\mu\left(\gamma_1(a_1-\gamma_2 b_1)-b_2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
x_{31} &= e^{(\gamma_1+\gamma_2)\,t}\left[-\kappa^2\omega_1^2\omega_2^2\theta\mu b_1+\kappa\lambda\left(\gamma_1(a_1+\gamma_2 b_1)+b_2\right)\right] \\
x_{32} &= e^{-(\gamma_1+\gamma_2)\,t}\left[\kappa\lambda(a_1+\gamma_2 b_1)-\kappa^2\omega_2^2\theta\left(a_1+(\gamma_1\mu+\gamma_2)b_1\right)\right] \\
x_{41} &= e^{(\gamma_1+\gamma_2)\,t}\left[-\omega_2^2\left(\kappa\omega_1^2\theta(a_1-\gamma_2 b_1)-\kappa^2\lambda(a_1-(\gamma_1\mu+\gamma_2)b_1)\right)\right] \\
x_{42} &= e^{-(\gamma_1+\gamma_2)\,t}\left[-\omega_2^2\left(\kappa^2\lambda\mu b_1+\kappa\theta(\gamma_1(a_1-\gamma_2 b_1)-b_2)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
x_{33} &= e^{-\gamma_2 t}\left[a_2+\gamma_2 b_2+\left(\gamma_1^2+2\gamma_1\gamma_2-\omega_1^2+\kappa\omega_1^2\omega_2^2\theta^2+\kappa\lambda^2\right)(a_1+\gamma_2 b_1)-\right. \\
&\quad \left.\kappa^2\omega_2^2(1-\mu)\left(a_1+(\gamma_1\mu+\gamma_2)b_1\right)\right] \\
x_{44} &= e^{\gamma_2 t}\left[a_2-\gamma_2 b_2+\left(\gamma_1^2+2\gamma_1\gamma_2-\omega_1^2+\kappa\omega_1^2\omega_2^2\theta^2+\kappa\lambda^2\right)(a_1-\gamma_2 b_1)-\right. \\
&\quad \left.\kappa^2\omega_2^2(1-\mu)\left(a_1-(\gamma_1\mu+\gamma_2)b_1\right)\right] \\
x_{43} &= e^{-\gamma_2 t}\left[-\omega_2^2\left(\kappa^2\lambda^2\mu b_1+\kappa\mu((\gamma_1^2+2\gamma_1\gamma_2-\omega_1^2)b_1+b_2)\right)\right] \\
x_{34} &= e^{\gamma_2 t}\left[\kappa^2\omega_1^2\omega_2^2\theta^2\mu b_1+\kappa\mu((\gamma_1^2+2\gamma_1\gamma_2-\omega_1^2)b_1+b_2)\right]
\end{aligned}
$$

Where

$$
a_1 = \frac{1}{w_3}\left(\cos(\Omega_1 t)-\cos(\Omega_2 t)\right) \qquad b_1 = \frac{1}{w_3}\left(\frac{\sin(\Omega_1 t)}{\Omega_1}-\frac{\sin(\Omega_2 t)}{\Omega_2}\right)
$$

$$
a_2 = \frac{1}{w_3}\left(\left(\gamma_2^2+\Omega_1^2\right)\cos(\Omega_1 t)-\left(\gamma_2^2+\Omega_2^2\right)\cos(\Omega_2 t)\right)
$$

$$
b_2 = \frac{1}{w_3}\left(\left(\gamma_2^2+\Omega_1^2\right)\frac{\sin(\Omega_1 t)}{\Omega_1}-\left(\gamma_2^2+\Omega_2^2\right)\frac{\sin(\Omega_2 t)}{\Omega_2}\right) \tag{17}
$$

The frequencies Ω_1 and Ω_2 are the solution of the following algebraic system

$$
w_1 = \left(\gamma_2^2+\Omega_1^2\right)\left(\gamma_2^2+\Omega_2^2\right) = \kappa^2\omega_2^2\mu\left(\omega_1^2\mu-\gamma_1(\gamma_1\mu+2\gamma_2)\right) \tag{18}
$$

$$
w_2 = (\gamma_2^2+\Omega_1^2)+(\gamma_2^2+\Omega_2^2) = \omega_1^2+\kappa^2\omega_2^2-\kappa(\omega_1^2\omega_2^2\theta^2+\lambda^2)-\gamma_1(\gamma_1+2\gamma_2) \tag{19}
$$

We have also set

$$
w_3 = \Omega_1^2-\Omega_2^2 = \sqrt{w_2^2-4w_1} \tag{20}
$$

We finally find the following solutions which are time independent.

$$
\Omega_1 = \pm\sqrt{(w_2+w_3)/2-\gamma_2^2} \qquad \Omega_2 = \pm\sqrt{(w_2-w_3)/2-\gamma_2^2} \tag{21}
$$

The above results are valid for any particular value of the parameters with the exception of the parameters λ and θ which must be real. The solutions (eqs. 17) have well defined limits when $\Omega_1 \to 0$ and $\Omega_2 \to 0$, or for the more interesting limit $\Omega_1 \to 0$ and $\Omega_2 \to \pm i\gamma_2$.
The results found in this paper, coincide with that of paper (31) when $\gamma_1 = 0$, where we have a constant magnetic field. For this zero value of parameter γ_1, equation (18) becomes symmetric, which means that it has two solutions with respect to μ, of opposite signs. This symmetry is present again if $\gamma_2 = 0$. This symmetry of the μ parameter is crucial, since it is the deformed parameter of the second phase space (equation 4) and the transformation $\mu \to -\mu$ means that $[\hat{Q}_2, \hat{P}_2] \to -[\hat{Q}_2, \hat{P}_2]$.
We can reconstruct the same symmetry also if we set

$$\gamma_1\mu + 2\gamma_2 = 0 \quad \Longrightarrow \quad \left(\gamma_2^2 + \Omega_1^2\right)\left(\gamma_2^2 + \Omega_1^2\right) = \kappa^2\omega_1^2\omega_2^2\mu^2 \tag{22}$$

In this case we find

$$\gamma_1 + \gamma_2 = \frac{2-\mu}{\mu}\gamma_2 = \frac{\hbar + \frac{\lambda\theta}{\hbar}}{\hbar - \frac{\lambda\theta}{\hbar}}\gamma_2 \tag{23}$$

This term which is on the exponential of the first Hamiltonian in eq. (6), changes sign in the duality $\hbar \to \lambda\theta/\hbar$. For all the above symmetric cases equation (18) has two solutions with respect to μ with opposite signs.
In general equation (18) is a parabola with respect to μ and becomes zero for the following two values of the deformed parameter μ.

$$\mu = 0 \;\Rightarrow\; \lambda = \frac{1}{\theta} \qquad \textit{and} \qquad \mu = \frac{2\gamma_1\gamma_2}{\omega_1^2 - \gamma_1^2} \;\Rightarrow\; \lambda = \frac{1}{\theta}\left(1 - \frac{2\gamma_1\gamma_2}{\omega_1^2 - \gamma_1^2}\right) \tag{24}$$

In both cases the frequencies are identical if $\omega_1^2 = \gamma_1^2 + \gamma_1\gamma_2$.
We will solve the case where $\mu = 0$ in the next paragraph because it leads to the relation $[\hat{Q}_2, \hat{P}_2] = 0$ which means that the problem is one dimensional since the commutator of the second phase space vanishes.
When $\omega_1 = \pm\gamma_1$ equation (18) has only one solution, with respect to μ, We find

$$\mu = -\frac{\left(\gamma_2^2 + \Omega_1^2\right)\left(\gamma_2^2 + \Omega_1^2\right)}{2\kappa^2\gamma_1\gamma_2\,\omega_2^2} \tag{25}$$

while the second solution tends to minus infinity.
Next we will calculate the exact propagator of the system.
As is well known the action of the time evolution operator on the delta function, produces the propagator of the system.

$$G(\tau_1, \tau_1', \tau_2, \tau_2', t) = \hat{U}(t)\delta(\tau_1 - \tau_1')\delta(\tau_2 - \tau_2') \tag{26}$$

Because of the commutation relations (10) only two quantities can be simultaneously measured. We choose the following observables

$$\hat{T}_2 = \hat{Q}_1 = q_1 \to \tau_1 \qquad \hat{T}_4 = \hat{Q}_2 = q_2 - \frac{\theta}{\hbar}e^{\gamma_1 t}p_1 \to \tau_2 \tag{27}$$

For the calculations we consider the following representation

$$\hat{T}_1 = -c_{21}\partial_{\tau_1} \qquad \hat{T}_2 = \tau_1 \qquad \hat{T}_3 = -c_{43}\partial_{\tau_2} \qquad \hat{T}_4 = \tau_2 \tag{28}$$

We can of course choose another couple of commuting observable. The various propagators are appropriate Fourier transforms of each other.
By the help of the above representation, and after a simple calculation (32) we find the propagator:

$$G(\tau_1, \tau'_1, \tau_2, \tau'_2, t) = \tag{29}$$

$$\frac{1}{\sqrt{s_0}} \exp\left\{ -\frac{1}{2s_0} \left[c_{43}(2\tau_1\tau'_1 x_{34} + \tau'^2_1(x_{14}x_{31} - x_{11}x_{34}) + \tau_1^2(x_{24}x_{32} - x_{22}x_{34})) + \right.\right.$$

$$c_{21}(2\tau_2\tau'_2 x_{12} + \tau'^2_2(x_{13}x_{32} - x_{12}x_{33}) + \tau_2^2(x_{14}x_{42} - x_{12}x_{44})) -$$

$$\left.\left. 2c_{21}\tau'_1(\tau_2 x_{14} - \tau'_2(x_{14}x_{33} - x_{13}x_{34})) - 2c_{43}\tau_1(\tau'_2 x_{32} + \tau_2(x_{34}x_{42} - x_{32}x_{44})) \right] \right\}$$

where

$$s_0 = c_{21}c_{43}\,(x_{12}x_{34} - x_{14}x_{32}) \tag{30}$$

If we find the propagator we can calculate the time evolution of a quantum system which must have initially the following form.

$$\psi(\tau_1, \tau_2, t)|_{t=0} = \psi_0(q_1, q_2 - \frac{\theta}{\hbar} e^{\gamma_1 t} p_1, 0) \tag{31}$$

We can not get rib of the time factor in the initial state except if $\gamma_1 = 0$ or $\theta = 0$, where we have a time independent non commutative space or an ordinary commutative space respectively. This is a consequence of the time dependance of the Bopp shift transformations (4), as the whole phase space of the second point is moving with respect to the first one.
The wave function of the system is given by the relation

$$\psi(\tau_1, \tau_2, t) = \iint G(\tau_1, \tau'_1, \tau_2, \tau'_2, t)\psi(\tau'_1, \tau'_2, 0) d\tau'_1 \tau'_2 \tag{32}$$

The Hamiltonian (6) has the same form as the Hamiltonian in ordinary quantum mechanics. As a consequence the spectrum of $\hat{\mathcal{H}}$ is a linear combination of the following energies

$$E_j = \hbar\Omega_j \left(n_j + \frac{1}{2} \right), \qquad n_j \in \mathcal{N} \tag{33}$$

The frequencies Ω_1 and Ω_2 are given by equations (21).

5. One dimensional case

In this section we will examine the case where we have one external frequency.

$$c_{43} = 0 \quad \Longrightarrow \quad \mu = 0 \quad \Longrightarrow \quad \frac{\lambda}{\hbar} = \frac{\hbar}{\theta} \tag{34}$$

The system is now one dimensional while the second Hamiltonian is like a potential energy. The functions f_{jk} satisfy a differential system which can be found in ref (32). The solution is as follows:

$$
\begin{aligned}
f_{11} &= -\frac{i}{2}\,\frac{1-\kappa\omega_2^2\theta^2}{\cos(\Omega\,t)-(\gamma_1+\gamma_2)\,\frac{\sin(\Omega\,t)}{\Omega}}\,\frac{\sin(\Omega\,t)}{\Omega}\\
f_{21} &= -i(\gamma_1+\gamma_2)\,t - i\log\left\{\cos(\Omega\,t)-(\gamma_1+\gamma_2)\frac{\sin(\Omega\,t)}{\Omega}\right\}\\
f_{22} &= -\frac{i}{2}\,e^{-2(\gamma_1+\gamma_2)\,t}\,\frac{\omega_1^2-\kappa\lambda^2}{\cos(\Omega\,t)-(\gamma_1+\gamma_2)\frac{\sin(\Omega\,t)}{\Omega}}\,\frac{\sin(\Omega\,t)}{\Omega}\\
f_{31} &= ie^{(\gamma_1+\gamma_2)\,t}\kappa\lambda\,\frac{1-\kappa\omega_2^2\theta^2}{\gamma_2^2+\Omega^2}\left\{e^{-\gamma_2 t}-\cos(\Omega\,t)+\gamma_2\frac{\sin(\Omega\,t)}{\Omega}\right\}\\
f_{32} &= -ie^{-(\gamma_1+\gamma_2)\,t}\kappa\lambda\left\{\frac{\sin(\Omega\,t)}{\Omega}+\frac{\gamma_1}{\gamma_2^2+\Omega^2}\left(e^{-\gamma_2 t}-\cos(\Omega\,t)+\gamma_2\frac{\sin(\Omega\,t)}{\Omega}\right)\right\}\\
f_{33} &= -\frac{i}{2}e^{-\gamma_2\,t}\,\frac{k}{\gamma_2^2+\Omega^2}\left\{(1-\kappa\omega_2^2\theta^2)\left(k\lambda^2\frac{\sin(\Omega\,t)}{\Omega}-\omega_1^2\frac{\sinh(\gamma_2\,t)}{\gamma_2}\right)+\right.\\
&\quad\left.\gamma_1(\gamma_1+2\gamma_2)\frac{\sinh(\gamma_2\,t)}{\gamma_2}\right\}-\frac{i}{2}f_{31}f_{32}\\
f_{41} &= ie^{(\gamma_1+\gamma_2)\,t}\kappa\omega_2^2\theta\left\{\frac{\sin(\Omega\,t)}{\Omega}-\frac{\gamma_1}{\gamma_2^2+\Omega^2}\left(e^{\gamma_2 t}-\cos(\Omega\,t)+\gamma_2\frac{\sin(\Omega\,t)}{\Omega}\right)\right\}\\
f_{42} &= ie^{-(\gamma_1+\gamma_2)\,t}\,\frac{\kappa\omega_2^2\theta(\omega_1^2-\kappa\lambda^2)}{\gamma_2^2+\Omega^2}\left(e^{\gamma_2 t}-\cos(\Omega\,t)+\gamma_2\frac{\sin(\Omega\,t)}{\Omega}\right)\\
f_{43} &= -i\frac{\kappa^2\omega_2^2}{\gamma_2^2+\Omega^2}\left\{\gamma_1\,t-e^{-\gamma_2\,t}\gamma_1\frac{\sin(\Omega\,t)}{\Omega}-\right.\\
&\quad\left.e^{-\gamma_2 t}\left(1+\frac{2\gamma_1\gamma_2}{\gamma_2^2+\Omega^2}\right)\left(e^{\gamma_2 t}-\cos(\Omega\,t)+\gamma_2\frac{\sin(\Omega\,t)}{\Omega}\right)\right\}\\
f_{44} &= \frac{i}{2}\,e^{\gamma_2 t}\,\frac{\kappa\omega_2^2}{\gamma_2^2+\Omega^2}\left\{(\omega_1^2-\kappa\lambda^2)\left(\frac{\sinh(\gamma_2\,t)}{\gamma_2}-\kappa\omega_2^2\theta^2\frac{\sin(\Omega\,t)}{\Omega}\right)-\right.\\
&\quad\left.\gamma_1(\gamma_1+2\gamma_2)\frac{\sinh(\gamma_2\,t)}{\gamma_2}\right\}-\frac{i}{2}f_{41}f_{42}
\end{aligned}
$$

Where the frequency is

$$\Omega = \sqrt{\omega_1^2+\kappa^2\omega_2^2-\kappa(\omega_1^2\omega_2^2\theta^2+\lambda^2)-(\gamma_1+\gamma_2)^2} \tag{35}$$

So all the formulas involved depend on one common frequency Ω.
The same frequencies can be found also for the two dimensional case of the previous paragraph if

$$\omega_1^2 = \gamma_1(\gamma_1+\gamma_2) \qquad or \qquad \gamma_2 = -\gamma_1+\frac{\omega_1^2}{\gamma_1} \tag{36}$$

For this value of the friction parameter γ_2 and from equations (18) and (19) we find

$$\mu = 2 \quad or \quad \lambda = -\frac{1}{\theta} \quad \Rightarrow \quad \Omega_1 = \Omega \qquad\qquad \Omega_2 = i\gamma_2 \tag{37}$$

It seems that for both cases, namely

$$\mu = 1 - \frac{\lambda\theta}{\hbar^2} = 0 \qquad \textit{and} \qquad 2 - \mu = 1 + \frac{\lambda\theta}{\hbar^2} = 0 \tag{38}$$

the final propagators depend on only one frequency. In these cases, which we will study in the next section, the commutator of the second phase space is $[\hat{Q}_2, \hat{P}_2] = 0$ and $[\hat{Q}_2, \hat{P}_2] = 2i\hbar$ respectively.
In the space of the $\hat{T}_j$ operators, we have now three commutative operators. We choose the following:

$$q_1 \;\rightarrow\; \tau_1 \qquad\qquad p_2 + \frac{\lambda}{\hbar} e^{-\gamma_1 t} q_1 \;\rightarrow\; \pi_2 \qquad\qquad q_2 - \frac{\theta}{\hbar} e^{\gamma_1 t} p_1 \;\rightarrow\; \tau_2 \tag{39}$$

To calculate the propagator we assume the representation

$$\hat{T}_1 = -c_{21}\partial_{\tau_1} \qquad \hat{T}_2 = \tau_1 \qquad \hat{T}_3 = \pi_2 \qquad \hat{T}_4 = \tau_2 \tag{40}$$

To find the propagator we calculate first the propagator in the first dimension. After some algebra, we find the distribution.

$$G_1(\tau_1, \tau_1', t) = \hat{U}_2 \hat{U}_1 \delta(\tau_1 - \tau_1') = \frac{e^{-\frac{1}{2}(\gamma_1+\gamma_2)\, t}}{\sqrt{2i(1-\kappa\omega_2^2\theta^2)}} \sqrt{\frac{\Omega}{\sin(\Omega\, t)}}$$

$$\exp\left\{ \frac{i}{2} \frac{e^{-(\gamma_1+\gamma_2)\, t}}{1-\kappa\omega_2^2\theta^2} \frac{\Omega}{\sin(\Omega\, t)} \left[e^{-(\gamma_1+\gamma_2)\, t} \left(\cos(\Omega\, t) + (\gamma_1+\gamma_2) \frac{\sin(\Omega\, t)}{\Omega} \right) \tau_1^2 + \right.\right.$$

$$\left.\left. e^{(\gamma_1+\gamma_2)\, t} \left(\cos(\Omega\, t) - (\gamma_1+\gamma_2) \frac{\sin(\Omega\, t)}{\Omega} \right) {\tau_1'}^2 - 2\tau_1\tau_1' \right] \right\} \tag{41}$$

The final propagator can be found by the action of the operator $\hat{U}_4\hat{U}_3\delta(\tau_2 - \tau_2')$ on this distribution $G_1(\tau_1, \tau_1', t)$. Because $c_{43} = 0$, the operator $\hat{U}_4\hat{U}_3\delta(\tau_2 - \tau_2')$ does not contain any operators with respect to the second dimension. Consequently a delta function remains in the final result.
After a simple calculation we find

$$G(\tau_1, \tau_1', \tau_2, \tau_2', \pi_2, t) = \hat{U}_4\hat{U}_3\delta(\tau_2 - \tau_2')\, G_1(\tau_1, \tau_1', t) = \tag{42}$$

$$e^{f_{44}\tau_2^2 + f_{33}\pi_2^2 + (f_{43} - i f_{32} f_{41})\pi_2\tau_2 + \tau_1(f_{32}\pi_2 + f_{42}\tau_2)}\, G_1\left(\tau_1 - i f_{31}\pi_2 - i f_{41}\tau_2, \tau_1', t\right) \delta(\tau_2 - \tau_2')$$

If we find the propagator we can calculate the time evolution of a quantum system which must initially have the following form.

$$\psi(\tau_1, \tau_2, \pi_2, t)|_{t=0} = \psi_0(q_1, q_2 - \frac{\theta}{\hbar} e^{\gamma_1 t} p_1, p_2 + \frac{\lambda}{\hbar} e^{-\gamma_1 t} q_1, 0) \tag{43}$$

The initial wave function depends on time unless $\gamma_1 = 0$ or even $\gamma_1 \neq 0$ but $\theta = 0$ and $\lambda = 0$.

The wave function of the system is given by the following single integral.

$$\psi(\tau_1,\tau_2,\pi_2,t)=\iint G(\tau_1,\tau_1',\tau_2,\tau_2',\pi_2,t)\psi(\tau_1',\tau_2',\pi_2,0)d\tau_1'd\tau_2' = e^{f_{44}\tau_2^2+f_{33}\pi_2^2} \tag{44}$$
$$e^{(f_{43}-if_{32}f_{41})\pi_2\tau_2+\tau_1(f_{32}\pi_2+f_{42}\tau_2)}\int G_1\left(\tau_1-if_{31}\pi_2-if_{41}\tau_2,\tau_1',t\right)\psi(\tau_1',\tau_2,\pi_2,0)d\tau_1'$$

The spectrum of the Hamiltonian $\hat{\mathcal{H}}$ is the following

$$E=\hbar\Omega\left(n+\frac{1}{2}\right),\qquad n\in\mathcal{N} \tag{45}$$

which now depend on one frequency Ω of equation (35).

6. Canonical density matrix

As is well known we can find the statistical distribution function from the propagator. The relation is

$$\rho(\tau_1,\tau_1',\tau_2,\tau_2',b)=G(\tau_1,\tau_1',\tau_2,\tau_2',-i\hbar b) \tag{46}$$

where $b=1/kT$, k is the Boltzman constant and T is the temperature. To find the partition function, we set $\tau_1=\tau_1'$ and $\tau_2=\tau_2'$ and then we integrate the distribution $\rho(\tau_1,\tau_2,b)$ with respect to τ_1 and τ_2. The partition function is as follows

$$z(b)=\iint_{\Re^2}G(\tau_1,\tau_1,\tau_2,\tau_2,-i\hbar b)d\tau_1 d\tau_2 \tag{47}$$

The problem is two dimensional. The coordinates q_1 and q_2 are operators which do not commute and its commutator must be time dependent too ($\sim\theta e^{\gamma_1 t}$). With a particular value of the parameter $\mu=0$, the problem becomes one dimensional and so the resulting formulas depend on one frequency. We can also find one frequency for the two dimensional case. From equation (18), it is obvious that being $\mu=0$ requires that one of the frequencies Ω_1 and Ω_2 must be equal to $i\gamma_2$. We choose $\Omega_2=i\gamma_2$.
We will find the statistical partition function for two district cases which both depend on only one common frequency. One of these cases is the one dimensional case where $\mu=0$ and the other is the two dimensional case where $\Omega_2=i\gamma_2$. To simplify the results we will study the particular case where $\omega_1=0$ and $\gamma_1=-\gamma_2$. For these two cases we have

$$\theta=+1/\lambda\qquad or\qquad \mu=0\qquad one-dimensional \tag{48}$$
$$\theta=-1/\lambda\qquad or\qquad \mu=2\qquad two-dimensional \tag{49}$$

where of course $\theta\neq 0$ and $\lambda\neq 0$.
The Hamiltonian takes now the most simple form and it is the following:

$$\hat{\mathcal{H}}(\hat{p}_1,\hat{q}_1,\hat{p}_2,\hat{q}_2)=\frac{1}{2}\hat{p}_1^2-\kappa\left(e^{-2\gamma_2 t}\frac{1}{2}\hat{p}_2^2+e^{2\gamma_2 t}\frac{1}{2}\omega_2^2\hat{q}_2^2\right) \tag{50}$$

The basic operators satisfy the commutators

$$[\hat{q}_1,\hat{q}_2]=\pm i(1/\lambda)\,e^{\gamma_2 t},\qquad [\hat{p}_1,\hat{p}_2]=i\lambda\,e^{-\gamma_2 t},\qquad [\hat{q}_j,\hat{p}_k]=i\hbar\delta_{jk} \tag{51}$$

The common frequency of the final results for both cases, is

$$\Omega=\sqrt{\omega_2^2-\kappa\lambda^2} \tag{52}$$

which is independent of the parameter γ_2.
For the case of one dimension $(\mu = 0)$ the partition function is that of an ordinary oscillator that is

$$z_1(b) = \frac{\Omega}{\sinh\left(\frac{\Omega b}{2}\right)} \int_{\mathcal{R}} \left\{ \lim_{\tau_2' \to \tau_2} G(\tau_2, \pi_2, b)\, \delta(\tau_2 - \tau_2') \right\} d\tau_2 \tag{53}$$

where $G(\tau_2, \pi_2, b)$ is a Gaussian type classical distribution function which we do not write as can be found easily from equation (42).
For the case of two dimensions $(\mu = 2)$ we find the following partition function.

$$z_2(b) = \frac{\Omega^2 + \gamma_2^2}{16\,\kappa\omega_2 \sin\left(\frac{\gamma_2 b}{2}\right)} \sqrt{\frac{\Omega\gamma_2}{\gamma_2 \cos\left(\frac{\gamma_2 b}{2}\right)\sinh\left(\frac{\Omega b}{2}\right) - \Omega\cosh\left(\frac{\Omega b}{2}\right)\sin\left(\frac{\gamma_2 b}{2}\right)}}$$
$$\frac{1}{\sqrt{\Omega(\gamma_2^2 - \kappa\lambda^2)\cos\left(\frac{\gamma_2 b}{2}\right)\sinh\left(\frac{\Omega b}{2}\right) - \gamma_2\,\kappa^2\omega_2^2 \cosh\left(\frac{\Omega b}{2}\right)\sin\left(\frac{\gamma_2 b}{2}\right)}} \tag{54}$$

For high temperatures, that is for a large value of the temperature parameters, $(b \to 0)$, the partition function has singularities on the points $b = 2n\pi/\gamma_2$, where n is an integer $n = \pm 1, \pm 2, \pm 3, \cdots$.
For low temperatures, that is $b \to \infty$, the hyperbolic functions $\cosh(b\Omega/2)$ and $\sinh(b\Omega/2)$ are both equal to $e^{b\Omega/2}/2$. The partition function is multiplied with the factor $e^{-(1/2)b\,\Omega}$ and so tends to zero. The final partition function has singularities for the following values of b:

$$b = \frac{2}{\gamma_2}\left(n\pi + \arctan\left(\frac{\gamma_2}{\Omega}\right)\right) \qquad b = \frac{2}{\gamma_2}\left(n\pi + \arctan\left(\frac{\Omega}{\gamma_2}\,\frac{\gamma_2^2 - \kappa\lambda^2}{\kappa^2\omega_2^2}\right)\right) \tag{55}$$

In the sequel we will study the case where the parameter Ω becomes zero.
If $\kappa = -1$ the common frequency Ω can not be zero unless $\omega_2 \to \pm i\lambda$. For $\kappa = 1$ and $\lambda = \pm\,\omega_2$ we find $\Omega = 0$. The energy of the system now comes exclusively from the varying electromagnetic field while the energy eigenvalues of the Hamiltonian vanish.
The first partition function becomes:

$$z_1(b) = \frac{c}{\sqrt{b}\,\omega_2} \int_{\mathcal{R}} \left\{ \lim_{\tau_1' \to \tau_1} \delta(\tau_1 - \tau_1') \right\} d\tau_1 \tag{56}$$

The limit in the above equation, as well as the limit in the equation (53) means that the system can not be localized in space.
The second partition function has a well defined limit for $\Omega \to 0$. We find

$$\lim_{\Omega \to 0} z_2(b) = \frac{\gamma_2^2}{16\,\kappa^2\,\omega_2^2 \sin\left(\frac{b\gamma_2}{2}\right)} \frac{1}{\sqrt{\sin\left(\frac{b\gamma_2}{2}\right)}} \frac{1}{\sqrt{\sin\left(\frac{b\gamma_2}{2}\right) - \frac{b\gamma_2}{2}\cos\left(\frac{b\gamma_2}{2}\right)}} \tag{57}$$

This last partition function has again singularities for $b = 2n\pi/\gamma_2$, where $n \in \mathcal{A}$ and in addition for some new values of the temperature $b = 1/kT$ which are the solutions of the equation $\tan(b\,\gamma_2/2) = b\,\gamma_2/2$.
With the help of the partition function $z(b)$, we can find all the thermodynamic properties of the system. The thermodynamic energy is given by the following formula:

$$< E > = -\frac{\partial}{\partial b} \log(z(b)) \tag{58}$$

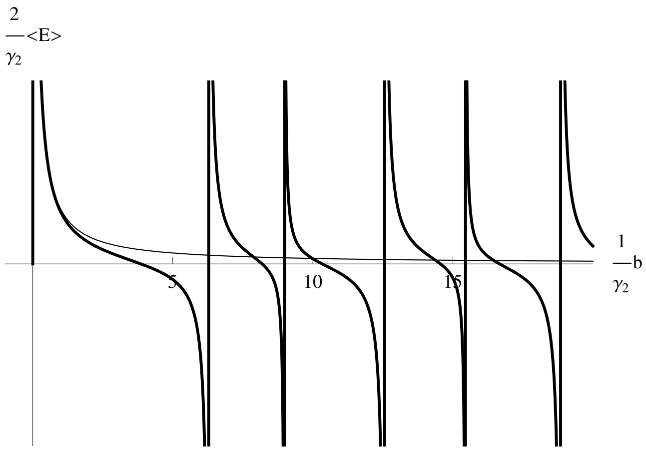

Fig. 2. The thermodynamic energy possesses many zeros and singularities.

After a simple calculation, and under certain conditions, we find the following thermodynamic energy

$$< E >= \frac{\gamma_2}{2} \frac{b\gamma_2 + 2b\gamma_2 \cos(b\gamma_2) - 3\sin(b\gamma_2)}{-2 + 2\cos(b\gamma_2) + b\gamma_2 \sin(b\gamma_2)} \tag{59}$$

which possesses many zeros and singularities. These particular points disappear if the parameter γ_2 vanishes. (Figure 2). We find

$$\lim_{\gamma_2 \to 0} < E >= 3/b \tag{60}$$

Noncommutativity is basically an internal geometric structure of the configuration space, which can not be observed per se. The last simple dynamical systems considered, have different configuration spaces and the same frequency which of course gives the same energy eigenvalues of the Hamiltonian. The thermodynamic singularities appears basically only for the two dimensional case and manifest the noncommutative phenomena. These singularities results from the varying magnetic field through the γ_2 parameter, and disappear for $\gamma_2 = 0$, that is for a constant magnetic field.

7. Conclusion

In this paper we have found the exact propagator of a two dimensional harmonic oscillator in non commutative quantum mechanics, where the ordinary non commutative parameters are time dependent. We have not investigated the problems which arise from this peculiar damping of the space, which stems from a time varying magnetic field. In this paper we are satisfied by the fact that the final results oscillate with two frequencies that are constant in time. The calculations have been made in such a way so that we can arrive at the final results, taking into consideration every special value of the parameters involved. We have first expand the time evolution operator in some kind of normal ordered form so that the propagator results easily by the action of this operator on some delta functions.

We have three damping mechanisms. The time dependent magnetic field $(\sim \lambda\, e^{\gamma_1 t})$, the main harmonic oscillator with varying mass $(\sim e^{-2(\gamma_1+\gamma_2)\, t})$ and the secondary harmonic oscillator with varying mass $\sim (e^{2\gamma_2\, t})$. The energy emitted from one of these is absorbed by the others so that the final results depend on two frequencies Ω_1 and Ω_2 which are time independent.
The momenta of the system satisfy a time dependent commutator $(\sim \lambda\, e^{-\gamma_1 t})$ which means that we have a time dependent magnetic field and consequently an electric field present. The coordinates of the system satisfy a commutator which is also time dependent $(\sim \theta\, e^{\gamma_1 t})$. The coordinates space is fuzzy and fluid with parameters θ and γ_1 respectively and the momenta is also fuzzy and fluid $(\lambda$ and $-\gamma_1)$.
With a linear time dependent transformation, the problem reduces to that of a two dimensional harmonic oscillator on a phase space with two independent phase spaces. The new commutator relations of this new phase space become time independent. The commutator between the coordinate and the momentum of the second phase space satisfies a deformed commutator relation equal to $i(\hbar - \lambda\theta/\hbar)$. This factor has no defined limit for $\hbar \to 0$, so the problem seems to have no classical analogy.
The Hamiltonian of the system is a linear combination of two Caldirola - Kanai Hamiltonians with friction parameters which differ by the parameter γ_1. H. Bateman uses a similar Hamiltonian and the energy emitted from one Hamiltonian was absorbed by the other one. In this paper the energy emitted from the first Hamiltonian on the point one and the time dependent magnetic field is absorbed by the other Hamiltonian on the point two. The time dependence of this second mirror Hamiltonian is the appropriate one, so that the resulting final formulas depend on two time independent frequencies.
The propagator provides the time evolution of the system. The initial wave functions depend on the two commuting variables (q_1, p_2) or (q_2, p_1). The first point particle has a well defined position and the second well defined momentum. The system can be considered as a massive object located at two separate massive points on a fuzzy and fluid two dimensional dynamical space.
This paper is a generalization of paper (31) in the case where the magnetic field is increasing (or decreasing) exponentially at a rate equal to γ_1. For $\gamma_1 = 0$ we found the same results while for $\gamma_1 \neq 0$ we derived some new interesting conclusions.
The parameter γ_1 destroys the time symmetry of the Hamiltonian and the product $m_1 m_2$ $(\sim e^{-2\gamma_1 t})$ is time increasing (or decreasing) exponentially. This is the main motivation of this paper. The propagator, as is well known, can also be used to find the statistical partition function and thus all the thermodynamical properties of the system. We have investigated two situations where the final results depend on only one common frequency Ω.
The first one is of course the case where the problem reduces to that of a one dimensional phase space, where $[\hat{Q}_2, \hat{P}_2] = 0$. The second is the two dimensional case where one of the frequencies becomes imaginary and equals to $\pm i\gamma_2$. We have investigated the case where $[\hat{Q}_2, \hat{P}_2] = 2i\hbar$. Because the oscillations give the energy of a harmonic oscillator the two cases have similar energies eigenvalues while they have different dimensionality. We have also found the statistical partition functions of these two systems. We have concluded that, in the two - dimensional case the partition function has many interesting zero's and singularities which are not present in the one - dimensional case. These particular points depend on the parameter γ_2, which is responsible for the time damping of the magnetic field.
Finally we have found the partition function in the resonance where this last frequency Ω becomes zero. The thermodynamic energy of the system, possesses similar zeros and singularities that disappear when γ_2 vanish.

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