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Quantum damped harmonic oscillator on non-commuting plane

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Abstract

In the present paper we study the quantum damped harmonic oscillator on non-commuting two-dimensional space. We calculate the time evolution operator and we find the exact propagator of the system. We investigate as well the thermodynamic properties of the system using the standard canonical density matrix. We find the statistical distribution function and the partition function. We calculate the specific heat for the limiting case of critical damping, where the frequencies of the system vanish. Finally we study the state of the system when the phase space of the second dimension becomes classical. We find that these systems have some singularities and zeros for low temperatures.

1. Introduction

The quantization of dissipative systems is of strong theoretical interest. The theory of a system with friction terms has been developed by many authors [1,2]. The damped harmonic oscillator has found many applications in quantum optics and plays a central role in the theory of lasers and masers. The effect of friction is to damp the free vibrations and so classically the oscillators are damped out in time.

In quantum mechanics there are many attempts to solve the problem. One method consists in using explicitly time-dependent Hamiltonian [3]. If the friction is a linear function of the velocity with friction constant γ the Hamiltonian takes the form

$$\mathscr{H} = e^{2\gamma t}\frac{1}{2m}\hat{p}^2 + e^{-2\gamma t}V(\hat{\vec{q}}) \tag{1}$$

known in the literature as the Caldirola–Kanai Hamiltonian. Some special potentials have been studied by several authors, especially the case of the harmonic oscillator [4,5]. The above Hamiltonian is not the energy or any constant of motion, but it is rather the generator of the motion. The usual quantization seems to violate the uncertainty principle [6].

Dissipation arises from interactions between the observed system and another one, often called the reservoir or the bath, into which energy flows in an irreversible manner. As a rule the system is embedded in some environment which is in principle unknown. A second group of methods for treating dissipative systems is based upon the procedure of doubling the phase space dimensions. H. Bateman to apply the usual canonical quantization method scheme has shown that one can double the numbers of degrees of freedom so as to deal

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with an effective isolated system. The new degrees of freedom may be assumed to represent the environment, which absorb the energy dissipated by the dissipative system [7]. The problem with this treatment lies on the fact that the two Hamiltonians do not commute [8].

In this paper, we will double the usual time-dependent Caldirola Hamiltonian of the harmonic oscillator of a system on a point (1)

$$\mathscr{H}_1(\hat{p}_1, \hat{q}_1, t) = \mathrm{e}^{-2\gamma t}\frac{1}{2m_1}\hat{p}_1^2 + \mathrm{e}^{2\gamma t}\frac{1}{2}m_1\omega_1^2 q_1^2 \tag{2}$$

to a second Hamiltonian to the point (2) which represents the environment

$$\mathscr{H}_2(\hat{p}_2, \hat{q}_2, t) = \mathrm{e}^{2\gamma t}\frac{1}{2m_2}\hat{p}_2^2 + \mathrm{e}^{-2\gamma t}\frac{1}{2}m_2\omega_2^2 q_2^2. \tag{3}$$

We choose the masses of the systems so that the product $m_1(t)m_2(t) = m_1 m_2$ becomes independent of time. So the effective frequencies $\widehat{\omega}_1$ and $\widehat{\omega}_2$ which we will find in the sequel become independent of the time as well. Our choice will be $m_1 = 1$ and $m_2 = 1$.

The Hamiltonian of the system will be the following linear combination:

$$\mathscr{H}(\hat{p}_1, \hat{q}_1, \hat{p}_2, \hat{q}_2, t) = \mathscr{H}_1(\hat{p}_1, \hat{q}_1, t) + \xi\mathscr{H}_2(\hat{p}_2, \hat{q}_2, t). \tag{4}$$

We will study the case $\xi = -1$. The Kanai–Caldirola Hamiltonian of two harmonic oscillators on non-commuting plane takes the form.

$$\mathscr{H} = \mathscr{H}_1(\hat{p}_1, \hat{q}_1, t) - \mathscr{H}_2(\hat{p}_2, \hat{q}_2, t) = \tfrac{1}{2}\mathrm{e}^{2\gamma t}(\hat{p}_1^2 - \omega_2^2\hat{q}_2^2) - \tfrac{1}{2}\mathrm{e}^{-2\gamma t}(\hat{p}_2^2 - \omega_1^2\hat{q}_1^2). \tag{5}$$

The Hamiltonian with $\xi = +1$, which describes a particle in a magnetic field, has been studied in Refs. [9,10]. Similar problems can be found in Refs. [11,12].

The Schrödinger equation of motion of the time evolution operator is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}U(t) = \{\mathscr{H}_1(\hat{p}_1, \hat{q}_1, t) - \mathscr{H}_2(\hat{p}_2, \hat{q}_2, t)\}U(t), \quad U(0) = 1. \tag{6}$$

The basic operators satisfy the commutation relations

$$[\hat{p}_1, \hat{p}_2] = \mathrm{i}\lambda, \quad [\hat{q}_1, \hat{q}_2] = \mathrm{i}\theta, \quad [\hat{q}_j, \hat{p}_k] = \mathrm{i}\hbar\delta_{jk}. \tag{7}$$

The first commutation relation expresses the existence of a magnetic field. The second commutation relation among coordinates represents the non-commutativity of space and the θ parameter has dimension of (length)^2. Spatial non-commutativity is completely consistent with the standard rules of quantum mechanics, but it implies the new Heisenberg relation $\Delta q_1 \Delta q_2 \sim \theta$ [13].

The system is a couple of two confluent particles or rather it is one massive object with some internal structure [14]. It is actually a damped harmonic oscillator coupled to its time-reserved image. There are no coupling terms in the Hamiltonian. The coupling comes from the nature of the non-commutative space. In non-commuting geometry we cannot say that we have two particles with masses $m_1\mathrm{e}^{2\gamma t}$ and $m_2\mathrm{e}^{-2\gamma t}$ located at the two separates points q_1 and q_2. If one of these have defined position, or momentum the position or the momentum of the other particle is quit indefinite and vice versa.

In the recent few years there is an increasing interesting in the non-commutative geometry [15] for the study of many physical problems. It becomes clear that there is a strong connection of these ideas with string theories and field theories [16] with many applications in solid state and particles physics [17].

In the present paper we will evaluate the exact propagator of the system. The paper is organized as follows.

We will first prove that the system is equivalent with a system in two dimensions. The first dimension is the usual one but the basic variables of the second extra dimension satisfy a deformed commutation relation.

In the next section we will find the exact propagator of the system. We will first expand the time evolution operator in an appropriate normal ordered form, so that the propagator can be calculated easily by the action of this operator on some delta functions.

Next we will investigate the thermodynamic properties of the system. We will find the standard canonical density matrix and the partition function. For zero friction the system is equivalent with that of a two-dimensional harmonic oscillator with two deformed frequencies $\widehat{\omega}_1$ and $\widehat{\omega}_2$.

In the next section we will find the partition function and the specific heat in the interesting limiting cases of critical dumping where the frequencies become zero. We will find the values of the temperature for which the partition function has singularities.

Finally we will investigate the system when the phase space of the second dimension is classical that is the position and the momentum commute. We will prove that the partition function of these states is equivalent with that of an one-dimensional system.

2. Two coupled systems

Any analytic function $g(\hat{p}_i, \hat{q}_i)$ of the basic operators $\hat{p}_i$ and $\hat{q}_i$ is defined by its power series expansion

$$g(\hat{p}_i, \hat{q}_i) = \sum_{j_1} \cdots \sum_{j_k} g(j_1, j_2, \ldots, j_k) \hat{p}_1^{j_1} \hat{q}_1^{j_2} \cdots \hat{p}_2^{j_n} \hat{q}_2^{j_k}. \tag{8}$$

We can use the commutation relations repeatedly to rearrange the operators $\hat{p}_i$ and $\hat{q}_i$, in order to write the function $f(\hat{p}_i, \hat{q}_i)$ in a desired form [18]. In this section we will find the normal ordered form of the time evolution operator. Because the Hamiltonian $\mathscr{H}(\hat{\vec{p}}, \hat{\vec{q}})$ is time dependent but quadratic in its arguments $\hat{\vec{p}}$ and $\hat{\vec{q}}$, this analysis is always possible.

We shall write the time evolution operator as a product of exponential operators of the form $\exp\{f_{jk}\hat{q}_j\hat{p}_k\}$ that is

$$U(t) = \prod_{jklmns} \mathrm{e}^{f_{jk}(t)\hat{q}_j\hat{q}_k} \mathrm{e}^{f_{lm}(t)\hat{q}_l\hat{p}_m} \mathrm{e}^{f_{ns}(t)\hat{p}_n\hat{p}_s}. \tag{9}$$

To find the unknown functions f_{jk} we differentiate the above operator with respect to time t and we find the operator

$$\left(\mathrm{i}\hbar \frac{\partial}{\partial t} U(t)\right) U(t)^{-1} = \mathscr{H}. \tag{10}$$

With the help of the operator relation

$$\mathrm{e}^{-bA} B \mathrm{e}^{bA} = B - b[A, B] + \frac{b^2}{2!}[A, [A, B]] + \cdots \tag{11}$$

we can write the above operator $\mathrm{i}\hbar \dot{U} U^{-1}$ as a polynomial of second order of coordinate and momenta operators. We equate this operator with the Hamiltonian which is a same order polynomial and finally we find a first-order differential system of the unknown functions $f_{jk}(t)$, which satisfy the initial conditions $f_{jk}(0) = 0$. If we find the functions f_{jk}, the propagator results easily from the equation.

$$G(q_1, q_1', q_2, q_2', t) = U(t)\delta(q_2 - q_2')\delta(q_1 - q_1'). \tag{12}$$

This method has been used by many authors [19,20].

We have a system of two independent oscillators with no coupling terms in the Hamiltonian. The coupling comes from the underline non-commutative structure of space. In this section we will transform the problem to that of two coupled harmonic oscillators in the usual quantum mechanical space. For this purpose we make the following linear transformations:

$$\begin{aligned} &\hat{P}_1 = \hat{p}_1, \quad \hat{Q}_1 = \hat{q}_1, \\ &\hat{P}_2 = \hat{p}_2 + \frac{\lambda}{\hbar}\hat{q}_1, \quad \hat{Q}_2 = \hat{q}_2 - \frac{\theta}{\hbar}\hat{p}_1. \end{aligned} \tag{13}$$

The commutators for the basic operators with the capital letters are

$$\begin{aligned} &[\hat{Q}_1, \hat{Q}_2] = 0, \quad [\hat{P}_1, \hat{P}_2] = 0, \\ &[\hat{Q}_1, \hat{P}_1] = \mathrm{i}\hbar, \quad [\hat{Q}_2, \hat{P}_2] = \mathrm{i}\hbar\left(1 - \frac{\lambda\theta}{\hbar^2}\right) = \mathrm{i}\hbar\mu. \end{aligned} \tag{14}$$

With these operators the Hamiltonian becomes

$$\mathscr{H}(\hat{\vec{P}},\hat{\vec{Q}},t)=\frac{\mathrm{e}^{2\gamma t}}{2}\left[\hat{P}_1^2-\omega_2^2\left(\hat{Q}_2+\hat{P}_1\frac{\theta}{\hbar}\right)^2\right]+\frac{\mathrm{e}^{-2\gamma t}}{2}\left[\omega_1^2\hat{Q}_1^2-\left(\hat{P}_2-\hat{Q}_1\frac{\lambda}{\hbar}\right)^2\right]. \tag{15}$$

This is a Hamiltonian of two coupled harmonic oscillators in a deformed quantum mechanical phase space. The deformation disappears (that is $\mu=1$) if one of the parameters λ or θ vanish. So in order to find the full effects of these parameters on the system we have to keep them both non-zero.

For the case where $\lambda=\hbar^2/\theta$ the μ parameter vanish and the operators $\hat{P}_2$ and $\hat{Q}_2$ commute. The problem is one-dimensional and the second dimension is actually an extra dimension [21].

In order to simplify the relations we shall use the following symbolism:

$$\hat{P}_1\to\hat{T}_1,\quad \hat{Q}_1\to\hat{T}_2,\quad \hat{P}_2\to\hat{T}_3,\quad \hat{Q}_2\to\hat{T}_4. \tag{16}$$

The commutation relations become

$$[\hat{T}_2,\hat{T}_1]=c_{21},\quad [\hat{T}_4,\hat{T}_3]=c_{43}=c_{21}\mu,\quad c_{21}=\mathrm{i}\hbar \tag{17}$$

while all the others commutators vanish. The Hamiltonian is written as

$$\mathscr{H}(\hat{\vec{T}})=k_{11}\hat{T}_1^2+k_{22}\hat{T}_2^2+k_{33}\hat{T}_3^2+k_{44}\hat{T}_4^2+k_{41}\hat{T}_4\hat{T}_1+k_{32}\hat{T}_3\hat{T}_2, \tag{18}$$

where we have set

$$\begin{aligned}&k_{11}=\frac{\mathrm{e}^{2\gamma t}}{2}\left(1-\omega_2^2\frac{\theta^2}{\hbar^2}\right),\quad k_{22}=\frac{\mathrm{e}^{-2\gamma t}}{2}\left(\omega_1^2-\frac{\lambda^2}{\hbar^2}\right),\\ &k_{33}=-\frac{\mathrm{e}^{-2\gamma t}}{2},\quad k_{44}=-\frac{\mathrm{e}^{2\gamma t}}{2}\omega_2^2,\quad k_{41}=-\mathrm{e}^{2\gamma t}\omega_2^2\frac{\theta}{\hbar},\quad k_{32}=\mathrm{e}^{-2\gamma t}\frac{\lambda}{\hbar}.\end{aligned} \tag{19}$$

The coupling terms k_{32} and k_{41} become zero in the case where $\lambda\to 0$ and $\theta\to 0$.

We will expand the time evolution operator in an appropriate ordered form so that the propagator will be calculated easily with a straight manner.

We will look for the following expansion:

$$U(t)=\mathrm{e}^{f_{44}T_4^2}\mathrm{e}^{(f_{43}/2)(T_4T_3+T_3T_4)}\mathrm{e}^{f_{42}T_4T_2}\mathrm{e}^{f_{41}T_4T_1}\mathrm{e}^{f_{33}T_3^2}\mathrm{e}^{f_{32}T_3T_2}\mathrm{e}^{f_{31}T_3T_1}\mathrm{e}^{f_{22}T_2^2}\mathrm{e}^{(f_{21}/2)(T_2T_1+T_1T_2)}\mathrm{e}^{f_{11}T_1^2}. \tag{20}$$

The operator is written as a product of four operators.

$$U(t)=U_4(\hat{T}_4,\hat{T}_3,\hat{T}_2,\hat{T}_1,t)U_3(\hat{T}_3,\hat{T}_2,\hat{T}_1,t)U_2(\hat{T}_2,\hat{T}_1,t)U_1(\hat{T}_1,t). \tag{21}$$

We will examine first the case where $c_{43}\neq 0$. Because of the commutation relations (17) only two quantities can be simultaneous measured. We choose the following observable:

$$T_2=Q_1=q_1\to\tau_1,\quad T_4=Q_2=q_2-\frac{\theta}{\hbar}p_1\to\tau_2. \tag{22}$$

For the calculations we consider the following standard representation:

$$T_1=-c_{21}\partial_{\tau_1},\quad T_2=\tau_1,\quad T_3=-c_{43}\partial_{\tau_2},\quad T_4=\tau_2. \tag{23}$$

We can of course choose another couple of two commuting observable. The various propagators are appropriate Fourier transforms of each other. If we find the propagator we can calculate the time evolution of the quantum system with the initial wave function $\Psi_0(p_1,q_2)$. The relation is

$$\Psi(\tau_1,\tau_2,t)=\iint G(\tau_1,\tau_1',\tau_2,\tau_2',t)\int \mathrm{e}^{-(\mathrm{i}/\hbar)p_1'\tau_1'}\Psi_0\left(p_1',\tau_2'+\frac{\theta}{\hbar}p_1'\right)\mathrm{d}p_1'\,\mathrm{d}\tau_1'\,\mathrm{d}\tau_2'. \tag{24}$$

The operator U_2U_1 depends only on the basic operators of the first dimension, so the propagator of the system is

$$G(\tau_2,\tau_2',\tau_1,\tau_1',t) = U_4U_3\delta(\tau_2-\tau_2')U_2U_1\delta(\tau_1-\tau_1'). \tag{25}$$

We denote with $x_{ij}(t)$ the functions which give the time evolution of the basic operators $\hat{T}_j$. That is

$$\hat{T}_i(t) = U(t)^{-1}\hat{T}_j(0)U(t) = x_{ij}(t)\hat{T}_j(0). \tag{26}$$

The unknown functions $x_{ij}(t)$ satisfy the differential system:

$$\begin{aligned} j=1,2,3,4,\quad & x_{1j}'(t) = -2k_{11}x_{2j}(t) + \mu k_{41}x_{3j}(t),\\ & x_{2j}'(t) = 2k_{22}x_{1j}(t) - \mu k_{32}x_{4j}(t),\\ & x_{3j}'(t) = k_{32}x_{1j}(t) - 2\mu k_{33}x_{4j}(t),\\ & x_{4j}'(t) = -k_{41}x_{2j}(t) + 2\mu k_{44}x_{3j}(t), \end{aligned} \tag{27}$$

with the initial condition $x_{ij}(0)=\delta_{ij}$.

We can prove that all the 10 unknown functions $f_{jk}(t)$ can be written with the help of the 16 functions $x_{jk}(t)$. We set

$$a_1 = \frac{x_{13}}{x_{33}},\quad a_2 = \frac{x_{23}}{x_{33}} \tag{28}$$

and we find

$$\begin{aligned} & f_{11} = -\frac{1}{2c_{21}}\frac{x_{12}-a_1x_{32}}{x_{11}-a_1x_{31}},\quad f_{21} = \frac{1}{c_{21}}\log(x_{11}-a_1x_{31}),\\ & f_{22} = \frac{1}{2c_{21}}\frac{x_{21}-a_2x_{31}}{x_{11}-a_1x_{31}},\quad f_{44} = \frac{1}{2c_{43}}\left(\frac{x_{43}}{x_{33}} - \frac{1}{\mu}\frac{x_{13}}{x_{33}}\frac{x_{23}}{x_{33}}\right),\\ & f_{41} = \frac{1}{c_{43}}x_{13},\quad f_{42} = \frac{1}{c_{43}}x_{23},\quad f_{43} = \frac{1}{c_{43}}\log x_{33},\\ & f_{33} = -\frac{1}{2c_{43}}\left(\frac{x_{34}}{x_{33}} + \frac{1}{\mu}(x_{14}-a_1x_{34})(x_{24}-a_2x_{34})\right),\\ & f_{31} = -\frac{1}{c_{43}}(x_{14}-a_1x_{34}),\quad f_{32} = -\frac{1}{c_{43}}(x_{24}-a_2x_{34}). \end{aligned} \tag{29}$$

With respect to the functions x_{jk} and if $f_{33}\neq 0$ and $f_{11}\neq 0$, the propagator finally becomes

$$\begin{aligned} & G(\tau_1,\tau_1',\tau_2,\tau_2',t)\\ & \quad = \frac{1}{\sqrt{s_0}}\exp\Bigg\{-\frac{1}{2s_0}[c_{43}(2\tau_1\tau_1'x_{34} + {\tau_1'}^2(x_{14}x_{31}-x_{11}x_{34}) + \tau_1^2(x_{24}x_{32}-x_{22}x_{34}))\\ & \qquad -2c_{43}\tau_1(\tau_2'x_{32} + \tau_2(x_{34}x_{42}-x_{32}x_{44})) - 2c_{21}\tau_1'(\tau_2x_{14} - \tau_2'(x_{14}x_{33}\\ & \qquad -x_{13}x_{34})) + c_{21}(2\tau_2\tau_2'x_{12} + {\tau_2'}^2(x_{13}x_{32}-x_{12}x_{33}) + \tau_2^2(x_{14}x_{42}-x_{12}x_{44}))]\Bigg\}, \end{aligned} \tag{30}$$

where $s_0 = c_{21}c_{43}(x_{12}x_{34}-x_{14}x_{32})$.

The calculation of all the formulas of this section can be found analytically in Ref. [22].

3. The propagator of the harmonic oscillator

In this section we will solve the system of differential equations (27). The solution is the following:

$$
\begin{aligned}
&x_{11} = e^{\gamma t}(a_3 - a_1\omega_2^2\mu), \quad x_{33} = e^{-\gamma t}(a_6 + a_2\omega_2^2\mu),\\
&x_{22} = e^{-\gamma t}(a_4 - a_2\omega_2^2\mu), \quad x_{44} = e^{\gamma t}(a_5 + a_1\omega_2^2\mu),\\
&x_{14} = \hbar^{-1}e^{\gamma t}a_1(\omega_2^2\theta - \lambda)\mu, \quad x_{32} = -\hbar^{-1}e^{-\gamma t}a_2(\omega_2^2\theta - \lambda),\\
&x_{23} = \hbar^{-1}e^{-\gamma t}a_2(\omega_1^2\theta - \lambda)\mu\omega_2^2, \quad x_{41} = -\hbar^{-1}e^{\gamma t}a_1(\omega_1^2\theta - \lambda)\omega_2^2,\\
&x_{13} = \hbar^{-1}e^{\gamma t}(a_7\lambda\mu - a_8\theta)\omega_2^2\mu, \quad x_{24} = \hbar^{-1}e^{-\gamma t}(a_7\omega_1^2\omega_2^2\theta\mu - a_8\lambda)\mu,\\
&x_{42} = -\hbar^{-1}e^{\gamma t}(a_7\lambda\mu - a_8\theta)\omega_2^2, \quad x_{31} = -\hbar^{-1}e^{-\gamma t}(a_7\omega_1^2\omega_2^2\theta\mu - a_8\lambda),\\
&x_{12} = e^{\gamma t}(a_7\omega_2^2\mu^2 - a_8(1 - \omega_2^2\theta^2\hbar^{-2})),\\
&x_{34} = -e^{-\gamma t}(a_7\omega_1^2(1 - \omega_2^2\theta^2\hbar^{-2})\mu - a_8),\\
&x_{21} = -e^{-\gamma t}(a_7\omega_1^2\omega_2^2\mu^2 - a_8(\omega_1^2 - \lambda^2\hbar^{-2})),\\
&x_{43} = e^{\gamma t}(a_7(\omega_1^2 - \lambda^2\hbar^{-2})\mu\omega_2^2 - a_8).
\end{aligned} \tag{31}
$$

The above functions a_j are

$$
\begin{aligned}
&a_1(\gamma) = \frac{1}{V}\left(\cos(\Omega_1 t) - \cos(\Omega_2 t) - \gamma\left(\frac{\sin(\Omega_1 t)}{\Omega_1} - \frac{\sin(\Omega_2 t)}{\Omega_2}\right)\right),\\
&a_3(\gamma) = \frac{1}{V}\left(\widehat{\omega}_1^2\cos(\Omega_1 t) - \widehat{\omega}_2^2\cos(\Omega_2 t) - \gamma\left(\widehat{\omega}_1^2\frac{\sin(\Omega_1 t)}{\Omega_1} - \widehat{\omega}_2^2\frac{\sin(\Omega_2 t)}{\Omega_2}\right)\right),\\
&a_5(\gamma) = \frac{1}{V}\left(\widehat{\omega}_1^2\cos(\Omega_2 t) - \widehat{\omega}_2^2\cos(\Omega_1 t) - \gamma\left(\widehat{\omega}_1^2\frac{\sin(\Omega_2 t)}{\Omega_2} - \widehat{\omega}_2^2\frac{\sin(\Omega_1 t)}{\Omega_1}\right)\right),\\
&a_2(\gamma) = a_1(-\gamma), \quad a_4(\gamma) = a_3(-\gamma), \quad a_6(\gamma) = a_5(-\gamma),\\
&a_7 = \frac{1}{V}\left(\frac{\sin(\Omega_1 t)}{\Omega_1} - \frac{\sin(\Omega_2 t)}{\Omega_2}\right),\\
&a_8 = \frac{1}{V}\left(\widehat{\omega}_1^2\frac{\sin(\Omega_1 t)}{\Omega_1} - \widehat{\omega}_2^2\frac{\sin(\Omega_2 t)}{\Omega_2}\right),
\end{aligned} \tag{32}
$$

where $V = \Omega_1^2 - \Omega_2^2$.

The effective frequencies $\widehat{\omega}_1, \widehat{\omega}_2$ and the final frequencies Ω_1, Ω_2 are

$$
\begin{aligned}
&\widehat{\omega}_1 = \frac{1}{2}\sqrt{(\omega_1 + \omega_2)^2 - \frac{1}{\hbar^2}(\lambda + \omega_1\omega_2\theta)^2} + \frac{1}{2}\sqrt{(\omega_1 - \omega_2)^2 - \frac{1}{\hbar^2}(\lambda - \omega_1\omega_2\theta)^2},\\
&\widehat{\omega}_2 = \frac{1}{2}\sqrt{(\omega_1 + \omega_2)^2 - \frac{1}{\hbar^2}(\lambda + \omega_1\omega_2\theta)^2} - \frac{1}{2}\sqrt{(\omega_1 - \omega_2)^2 - \frac{1}{\hbar^2}(\lambda - \omega_1\omega_2\theta)^2},\\
&\Omega_1 = \sqrt{\widehat{\omega}_1^2 - \gamma^2}, \quad \Omega_2 = \sqrt{\widehat{\omega}_2^2, -\gamma^2}.
\end{aligned} \tag{33}
$$

The effective frequencies $\widehat{\omega}_1$ and $\widehat{\omega}_2$, depend on the product $m_1m_2 = 1$. The deformed parameter μ, which we emphasize that it must be real, can be written as a function of the frequencies. We find

$$
\mu = \frac{\widehat{\omega}_1}{\omega_1}\frac{\widehat{\omega}_2}{\omega_2} = \frac{\sqrt{\Omega_1^2 + \gamma^2}\sqrt{\Omega_2^2 + \gamma^2}}{\omega_1\omega_2}. \tag{34}
$$

For the limit where $\omega_1 \to 0$ and $\omega_2 \to 0$ the above frequencies become $\widehat{\omega}_2 = 0$ and $\widehat{\omega}_1 = i\lambda/\hbar$, which means that we have squeeze states [23]. We will call the states where the effective frequencies $\widehat{\omega}_1$ and $\widehat{\omega}_2$ are imaginary squeezed states. Notice that for these states, the final frequencies Ω_1 and Ω_2 are also imaginary for every real value of γ and the deformed parameter μ is negative. For the limit where $\lambda \to 0$ and $\theta \to 0$ we find $\widehat{\omega}_1 = \omega_1$, $\widehat{\omega}_2 = \omega_2$ and $\mu = 1$.

We substitute the solutions (31) in the propagator (30) and after some algebra we finally find the desired propagator

$$\begin{aligned}
&G(\tau_1, \tau_1', \tau_2, \tau_2', t) \\
&= \frac{1}{\hbar\sqrt{d_0}} \frac{V}{\omega_1 \mu} \exp\left\{ -\frac{\mathrm{i}}{\hbar d_0} \frac{V^2}{\omega_1^2 \mu^2} \left[\omega_1 \mu \sqrt{c_1 c_2}((a_1 \tau_1' + a_2 \mathrm{e}^{-\gamma t} \tau_1 (a_5 + a_1 \omega_2^2 \mu)) \mathrm{e}^{\gamma t} \tau_2 \right.\right. \\
&\quad - (a_1 \tau_1' (a_6 + a_2 \omega_2^2 \mu) + a_2 \mathrm{e}^{-\gamma t} \tau_1) \tau_2') + \frac{\widehat{\omega}_1^2 \widehat{\omega}_2^2}{V^2} \left(c_1 \frac{\sin \Omega_1 t}{\Omega_1} - c_2 \frac{\sin \Omega_2 t}{\Omega_2} \right) \\
&\quad \times \left((\widehat{\omega}_1^2 \theta - \lambda \mu) \frac{\sin \Omega_1 t}{\Omega_1} - (\widehat{\omega}_2^2 \theta - \lambda \mu) \frac{\sin \Omega_2 t}{\Omega_2} \right) \frac{1}{\hbar} (\tau_1 \tau_2 - \tau_1' \tau_2') \\
&\quad + \frac{\omega_1^2 \mu^2}{2V} \mathrm{e}^{-\gamma t} \left(c_1 ((a_4 - a_2 \widehat{\omega}_1^2) \mathrm{e}^{-\gamma t} \tau_1^2 - 2\tau_1 \tau_1' + (a_3 - a_1 \widehat{\omega}_1^2) \mathrm{e}^{\gamma t} {\tau_1'}^2) \frac{\sin \Omega_1 t}{\Omega_1} \right. \\
&\quad \left. - c_2 ((a_4 - a_2 \widehat{\omega}_2^2) \mathrm{e}^{-\gamma t} \tau_1^2 - 2\tau_1 \tau_1' + (a_3 - a_1 \widehat{\omega}_2^2) \mathrm{e}^{\gamma t} {\tau_1'}^2) \frac{\sin \Omega_2 t}{\Omega_2} \right) \\
&\quad + \frac{1}{2V} \mathrm{e}^{\gamma t} \left(\widehat{\omega}_1^2 c_2 ((a_5 + a_1 \widehat{\omega}_2^2) \mathrm{e}^{\gamma t} \tau_2^2 - 2\tau_2 \tau_2' + (a_6 + a_2 \widehat{\omega}_2^2) e^{-\gamma t} {\tau_2'}^2) \frac{\sin \Omega_1 t}{\Omega_1} \right. \\
&\quad \left.\left.\left. - \widehat{\omega}_2^2 c_1 ((a_5 + a_1 \widehat{\omega}_1^2) \mathrm{e}^{\gamma t} \tau_2^2 - 2\tau_2 \tau_2' + (a_6 + a_2 \widehat{\omega}_1^2) \mathrm{e}^{-\gamma t} {\tau_2'}^2) \frac{\sin \Omega_2 t}{\Omega_2} \right) \right] \right\},
\end{aligned} \tag{35}$$

where

$$d_0 = 2c_1 c_2 (\cos \Omega_1 t \cos \Omega_2 t - 1) + (\widehat{\omega}_2^2 c_1^2 - 2\gamma^2 c_1 c_2 + \widehat{\omega}_1^2 c_2^2) \frac{\sin \Omega_1 t}{\Omega_1} \frac{\sin \Omega_2 t}{\Omega_2}. \tag{36}$$

The constants c_1 and c_2 are

$$c_1 = 1 - \omega_2^2 \frac{\theta^2}{\hbar^2} - \frac{\widehat{\omega}_1^2}{\omega_1^2}, \quad c_2 = 1 - \omega_2^2 \frac{\theta^2}{\hbar^2} - \frac{\widehat{\omega}_2^2}{\omega_1^2}. \tag{37}$$

The propagator is a two-dimensional Gaussian type distribution function of the variables $\tau_1 = q_1$ and $\tau_2 = q_2 - (\theta/\hbar) p_1$.

4. Canonical density matrix

As is well known we can find the statistical distribution function from the propagator. The relation is

$$\rho(\tau_1, \tau_1', \tau_2, \tau_2', b) = G(\tau_1, \tau_1', \tau_2, \tau_2', -\mathrm{i}\hbar b). \tag{38}$$

The special case where $\tau_1 = \tau_1'$ and $\tau_2 = \tau_2'$, Eq. (35) gives

$$\begin{aligned}
\rho(\tau_1, \tau_2, b) = \frac{\Omega_1^2 - \Omega_2^2}{\hbar \omega_1 \mu \sqrt{d_0}} \exp\left\{ \frac{\Omega_1^2 - \Omega_2^2}{\hbar d_0} \left[\frac{1}{\omega_1^2 \mu^2} \mathrm{e}^{-\mathrm{i}\hbar\gamma b} \tau_2^2 (\widehat{\omega}_2^2 d_1 c_1 - \widehat{\omega}_1^2 d_2 c_2) \right.\right. \\
\left.\left. + \mathrm{e}^{\mathrm{i}\hbar\gamma b} \tau_1^2 (d_1 c_2 - d_2 c_1) + \frac{2}{\omega_1 \mu} d_3 \sqrt{c_1 c_2} \tau_1 \tau_2 \right] \right\},
\end{aligned} \tag{39}$$

where we have set

$$\begin{aligned}
d_0 &= 2c_1c_2(\cosh(\hbar\Omega_1 b)\cosh(\hbar\Omega_2 b) - 1) - (\widehat{\omega}_2^2 c_1^2 - 2\gamma^2 c_1 c_2 + \widehat{\omega}_1^2 c_2^2)\frac{\sinh(\hbar\Omega_1 b)}{\Omega_1}\frac{\sinh(\hbar\Omega_2 b)}{\Omega_2},\\
d_1 &= \frac{\sinh(\hbar\Omega_2 b)}{\Omega_2}\left(\cos(\hbar\gamma b)\cosh(\hbar\Omega_1 b) + \gamma\sin(\hbar\gamma b)\frac{\sinh(\hbar\Omega_1 b)}{\Omega_1} - 1\right),\\
d_2 &= \frac{\sinh(\hbar\Omega_1 b)}{\Omega_1}\left(\cos(\hbar\gamma b)\cosh(\hbar\Omega_2 b) + \gamma\sin(\hbar\gamma b)\frac{\sinh(\hbar\Omega_2 b)}{\Omega_2} - 1\right),\\
d_3 &= \gamma\left((\cos(\hbar\gamma b) - \cosh(\hbar\Omega_1 b))\frac{\sinh(\hbar\Omega_2 b)}{\Omega_2} - (\cos(\hbar\gamma b) - \cosh(\hbar\Omega_2 b)\frac{\sinh(\hbar\Omega_1 b)}{\Omega_1}\right)\\
&\quad + (\cosh(\hbar\Omega_1 b) - \cosh(\hbar\Omega_2 b)\sin(\hbar\gamma b). \qquad (40)
\end{aligned}$$

We integrate the distribution $\rho(\tau_1, \tau_2, b)$ with respect to τ_1 and τ_2 and under certain conditions, we find the following partition function:

$$Z(b) = \frac{\sqrt{d_0}}{2\sqrt{(\widehat{\omega}_2^2 d_1 c_1 - \widehat{\omega}_1^2 d_2 c_2)(d_1 c_2 - d_2 c_1) - d_3^2 c_1 c_2}}. \qquad (41)$$

Because of the terms $e^{i\hbar\gamma b}$ and $e^{-i\hbar\gamma b}$ which appears in the exponential of the $\rho(\tau_1, \tau_2, b)$ distribution, the integral may not converge for some values of b. As a consequence the expression under the root of Eq. (41) may be negative. To avoid this unacceptable fact, the contour of integration has to be deformed from Euclidean to complex metrics in order to make the integral converge.

The exponent of the ρ distribution function contains the complex term

$$e^{i\hbar\gamma b}\tau_1^2 \quad \text{and} \quad e^{-i\hbar\gamma b}\tau_2^2.$$

So if the term under the square root of the partition function is positive we integrate on the contours y_1 and y_2 where

$$y_1 = e^{i\hbar\gamma b}\tau_1, \quad y_2 = e^{-i\hbar\gamma b}\tau_2 \implies y_1 y_2 = q_1 q_2 \qquad (42)$$

while if this term becomes negative, the contour must be

$$y_1 = ie^{i\hbar\gamma b}\tau_1, \quad y_2 = ie^{-i\hbar\gamma b}\tau_2 \implies y_1 y_2 = -q_1 q_2. \qquad (43)$$

The partition function has a denumerable number of singularities, which comes from the trigonometric terms $\cos(\hbar\gamma b)$ and $\sin(\hbar\gamma b)$ of the denominator. The argument of these terms is analogous to the friction constant γ and so these singularities disappear in the limit of zero friction $\gamma = 0$. The limit $\gamma \to 0$ gives the partition function of two independent harmonic oscillators with frequencies $\Omega_1 = \widehat{\omega}_1$ and $\Omega_2 = \widehat{\omega}_2$. We find

$$Z(b) = \frac{1}{4\sinh(\hbar\widehat{\omega}_1 b/2)\sinh(\hbar\widehat{\omega}_2 b/2)}. \qquad (44)$$

Notice that none of the effective frequencies $\widehat{\omega}_1$ or $\widehat{\omega}_2$ can be zero, because of relation (34) which implies that $\mu = 0$.

As in the classical treatment of the problem the solution have actually three kind of solutions: (a) $\Omega_1 \in \mathscr{R}$ and $\Omega_2 \in \mathscr{R}$, (b) $\Omega_1 \in \mathscr{C}$ and $\Omega_2 \in \mathscr{R}$ and (c) $\Omega_1 \in \mathscr{C}$ and $\Omega_2 \in \mathscr{C}$. For the last two cases, where one or two of the Ω's become imaginary, the hyperbolic functions in the term d_0 become trigonometric. So the function d_0 and consequently the whole partition function becomes zero for some values of b.

5. The critical damping

The above formulas have well-defined limits when the frequencies Ω_1 and Ω_2 become zero. This can be achieved with the following values of the parameters:

$$\omega_1 = \pm\omega_2 = \frac{\gamma}{\sqrt{\mu}} = \sqrt{\frac{\lambda}{\theta}} = \sigma, \quad \gamma = \sqrt{\frac{\lambda}{\theta}}\sqrt{\mu}, \quad \mu = 1 - \frac{\lambda\theta}{\hbar^2} = \frac{\gamma^2}{\omega_1\omega_2}. \tag{45}$$

In this limit of zero frequency oscillators, the partition function becomes

$$Z(b) = \frac{1}{2(1-\cos(\hbar\gamma b) - \hbar\gamma b\sin(\hbar\gamma b))}. \tag{46}$$

We set $b = 1/kT$ where k is the Boltzmann constant and T is the temperature and we find the following partition function:

$$Z(T) = \frac{1}{4\sin(\hbar\gamma/2kT)(\sin(\hbar\gamma/2kT) - \hbar\gamma/kT\cos(\hbar\gamma/2kT))}. \tag{47}$$

This function has some singularities and tends to infinity for a denumerable number of the temperature T. The denominator of the partition function becomes zero for the first time as T goes to 0, for

$$T \cong 0.428978\frac{\hbar\gamma}{k} = 0.428978\frac{\hbar}{k}\sqrt{\frac{\lambda}{\theta}}\sqrt{1-\frac{\lambda\theta}{\hbar^2}}. \tag{48}$$

Below this value it becomes negative and positive periodically. The values of T where the denominator of the partition function becomes zero, are

$$T_n \cong \frac{\hbar\gamma}{2k}\frac{1}{1.16556+n\pi}, \; n = 0,1,2,\ldots, \quad T_n = \frac{\hbar\gamma}{2k}\frac{1}{n\pi}, \; n = 1,2,\ldots. \tag{49}$$

Because we have again the problem of negative values we define the free energy of the system as

$$F = kT\log\left\{\pm 2\left(1-\cos\left(\frac{\hbar\gamma}{kT}\right) - \frac{\hbar\gamma}{kT}\sin\left(\frac{\hbar\gamma}{kT}\right)\right)\right\}. \tag{50}$$

We can say that the formula with the positive sign comes from the main system, while that with the negative sign comes from its mirror environment.

The specific heat $c = -T\,\partial^2F/\partial T^2$ is as follows:

$$c = 2k\frac{T_0^2}{T^2}\left(2\frac{T_0^2}{T^2} - 1 + 2\cos\left(\frac{T_0}{T}\right) - \cos\left(2\frac{T_0}{T}\right) - 2\frac{T_0}{T}\sin\left(\frac{T_0}{T}\right)\right)Z^2, \tag{51}$$

where

$$T_0 = \frac{\hbar}{k}\gamma = \frac{\hbar}{k}\sqrt{\frac{\lambda}{\theta}}\sqrt{1-\frac{\lambda\theta}{\hbar^2}} \tag{52}$$

is the Debye temperature. The parameters λ and θ have the same sign and so the temperature T_0 is imaginary for $\mu<0$ and it is real for $\mu>0$.

For the classical limit $\hbar \to 0$ or for large values of $\theta \to \infty$, we have

$$1 \ll \frac{\lambda\theta}{\hbar^2} \Longrightarrow T_0 \to \frac{\hbar}{k}\sqrt{\frac{\lambda}{\theta}}\sqrt{-\frac{\lambda\theta}{\hbar^2}} = \mathrm{i}\frac{\lambda}{k}. \tag{53}$$

The temperature T_0 is independent of θ and $\hbar$ and in addition it is imaginary. So the specific heat has no singularities (Fig. 1).

For small values of $\lambda\theta$ we have

$$1 \gg \frac{\lambda\theta}{\hbar^2} \Longrightarrow T_0 \to \frac{\hbar}{k}\sqrt{\frac{\lambda}{\theta}} = \frac{\hbar}{k}\sigma. \tag{54}$$

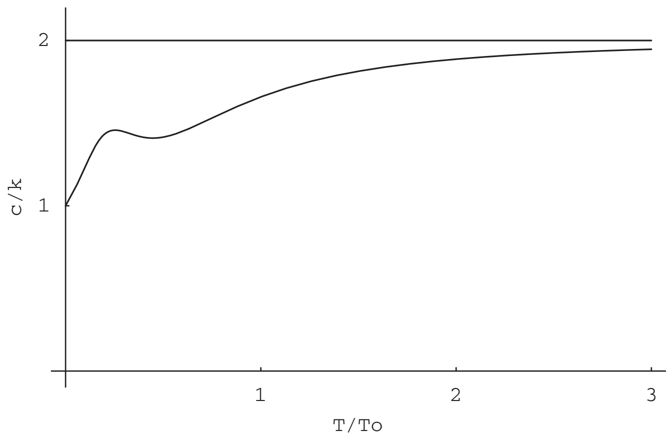

Fig. 1. Critical dumping. The specific heat for $T_0 = i$.

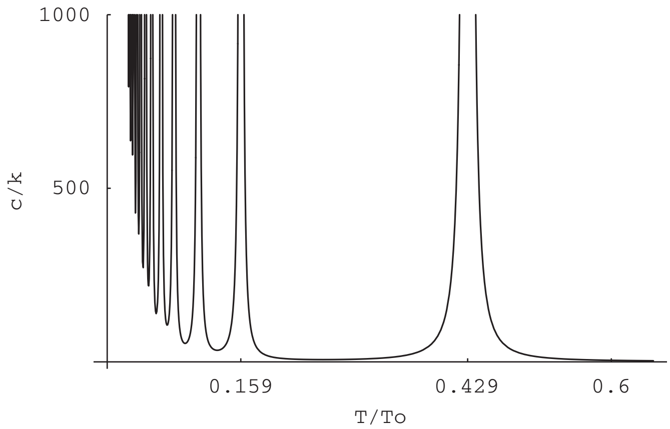

Fig. 2. Critical dumping. The specific heat for $T_0 = 1$.

The temperature T_0 is real and depends on the ratio λ/θ. The specific heat possesses a denumerable number of singularities for low temperature [24] (Fig. 2). The specific heat for both cases tends to the value 2 for $T \to \infty$.

6. The case where $\mu = 0$

We will find in this section the propagator of the system, for the case where the basic operators with the capital letters $\hat{P}_2$ and $\hat{Q}_2$ commute that is $\mu = 0$ or $\lambda = \hbar^2/\theta$. The propagator can be written with the help of the following parameters and frequencies:

$$\hbar = \sqrt{\lambda\theta}, \quad \sigma = \sqrt{\frac{\lambda}{\theta}}, \quad \widehat{\omega} = \frac{1}{\sigma}\sqrt{(\sigma^2 - \omega_2^2)(\omega_1^2 - \sigma^2)}, \quad \Omega = \sqrt{\widehat{\omega}^2 - \gamma^2}. \tag{55}$$

The effective frequency $\widehat{\omega}$ is zero for $\sigma = \omega_1$ or ω_2. It is real if $\omega_2 < \sigma < \omega_1$ otherwise or even if $\omega_2 = \omega_1$ it is imaginary.

With the help of the following functions:

$$a_9(\gamma) = e^{-\gamma t}\left(\cos(\Omega t) + \gamma\frac{\sin(\Omega t)}{\Omega}\right), \quad a_0(\gamma) = e^{\gamma t}\left(\cos(\Omega t) - \gamma\frac{\sin(\Omega t)}{\Omega}\right) \tag{56}$$

and after some calculations we find the following propagator:

$$\begin{aligned}
&G(\tau_1,\tau_1',\tau_2,\tau_2',\pi_2,t)\\
&=\frac{\sqrt{\sigma^2\Omega\mathrm{e}^{-\gamma t}}}{\sqrt{2\mathrm{i}\hbar(\sigma^2-\omega_2^2)\sin(\Omega t)}}\exp\Bigg\{-\frac{\mathrm{i}}{2\hbar}\Bigg[\frac{2\sigma\omega_2^2}{\omega_2^2-\sigma^2}\tau_2(\tau_1-\tau_1')\\
&+\frac{\sinh(\gamma t)}{\gamma}\left(\frac{\omega_1^2\mathrm{e}^{-\gamma t}}{\sigma^2-\omega_1^2}\pi_2^2+\frac{\sigma^2\omega_2^2\mathrm{e}^{\gamma t}}{\omega_2^2-\sigma^2}\tau_2^2\right)+\frac{\sigma^2\Omega\mathrm{e}^{-\gamma t}}{\sin(\Omega t)}\Bigg(\frac{a_9\tau_1^2-2\tau_1\tau_1'+a_0{\tau_1'}^2}{\omega_2^2-\sigma^2}\\
&+\frac{2\pi_2}{\sigma\widehat{\omega}^2}((1-a_9)\tau_1+(1-a_0)\tau_1')-\frac{\omega_2^2-\sigma^2}{\sigma^2\widehat{\omega}^4}(2-a_9-a_0)\pi_2^2\Bigg)\Bigg]\Bigg\}\delta(\tau_2-\tau_2'). \qquad (57)
\end{aligned}$$

The propagator is again a Gaussian type distribution but now it is a function of the three variables $\tau_1=\hat{q}_1$, $\tau_2=\hat{q}_2-(\theta/\hbar)\hat{p}_1$ and $\pi_2=\hat{p}_2+(\lambda/\hbar)\hat{q}_1$. The initial wave function must be a function of these three variables $\psi(\tau_1,\tau_2,\pi_2,t=0)$.

The statistical distribution function is given by the formula

$$\begin{aligned}
\rho(\tau_1,\tau_2,\pi_2,b)=&\frac{\sqrt{\sigma^2\Omega\mathrm{e}^{\mathrm{i}\hbar\gamma b}}}{\sqrt{2\hbar(\sigma^2-\omega_2^2)\sinh(\hbar\Omega b)}}\exp\Bigg\{\frac{\sin(\hbar\gamma b)}{2\hbar\gamma}\left(\frac{\omega_1^2\mathrm{e}^{\mathrm{i}\hbar\gamma b}}{\omega_1^2-\sigma^2}\pi_2^2-\frac{\sigma^2\omega_2^2\mathrm{e}^{-\mathrm{i}\hbar\gamma b}}{\omega_2^2-\sigma^2}\tau_2^2\right)\\
&+\frac{1}{2\hbar}\frac{\sigma^2\Omega\mathrm{e}^{\mathrm{i}\hbar\gamma b}(2-a_9-a_0)}{(\omega_2^2-\sigma^2)\sinh(\hbar\Omega b)}\left(\frac{\sigma\pi_2}{\sigma^2-\omega_1^2}-\tau_1\right)^2\Bigg\}\delta(\tau_2-\tau_2'). \qquad (58)
\end{aligned}$$

With the same as before conditions the integration with respect to τ_1 and τ_2 gives the partition function. We find

$$Z(b)=\frac{1}{\sqrt{2}\sqrt{1-\cosh(\hbar\Omega b)\cos(\hbar\gamma b)-(\gamma/\Omega)\sinh(\hbar\Omega b)\sin(\hbar\gamma b)}}, \qquad (59)$$

where we have set the integral with respect to τ_2 equal to one.

$$\int_{-\infty}^{\infty}\mathrm{d}\tau_2\left[\lim_{\tau_2'\to\tau_2}\delta(\tau_2-\tau_2')\exp\left\{\frac{\sin(\hbar\gamma b)}{2\hbar\gamma}\left(\frac{\omega_1^2\mathrm{e}^{\mathrm{i}\hbar\gamma b}}{\omega_1^2-\sigma^2}\pi_2^2-\frac{\omega_2^2\mathrm{e}^{-\mathrm{i}\hbar\gamma b}}{\omega_2^2-\sigma^2}\sigma^2\tau_2^2\right)\right\}\right]=1. \qquad (60)$$

We can write the above exponential with the standard form of a classical distribution function that is $\exp\{-b\mathscr{H}(\pi_2,\tau_2)\}$, if we set

$$\mathscr{H}(\pi_2,\tau_2)=\mathrm{e}^{\mathrm{i}\hbar\gamma b}\frac{1}{2M}\pi_2^2+\mathrm{e}^{-\mathrm{i}\hbar\gamma b}\frac{1}{2}M\Omega_0^2\tau_2^2, \qquad (61)$$

where

$$M=\frac{\hbar\gamma b}{\sin(\hbar\gamma b)}\left(\frac{\sigma_1^2}{\omega_1^2}-1\right),\quad \Omega_0=\frac{\omega_1\omega_2}{\widehat{\omega}}. \qquad (62)$$

This mass is zero for $\sigma=\omega_1$. In the classical limit where $\hbar\to 0$ which implies that $\lambda\to 0$, this mass is equal to -1.

The specific heat for $b=1/kT$ and $\hbar=1$ of this system is as follows:

$$\begin{aligned}
c=k\frac{\gamma^2+\Omega^2}{T^2}&\left(-1-\frac{\gamma^2}{\Omega^2}-\cos\left(\frac{2\gamma}{T}\right)+2\cos\left(\frac{\gamma}{T}\right)\cosh\left(\frac{\Omega}{T}\right)\right.\\
&\left.+\frac{\gamma^2}{\Omega^2}\cosh\left(\frac{2\Omega}{T}\right)-2\frac{\gamma}{\Omega}\sin\left(\frac{\gamma}{T}\right)\sinh\left(\frac{\Omega}{T}\right)\right)Z^4. \qquad (63)
\end{aligned}$$

For real γ and imaginary values of Ω this function has some singularities and also some zeros. (Fig. 3). For imaginary γ and real Ω the specific heat becomes zero for some value of the temperature T (Fig. 4).

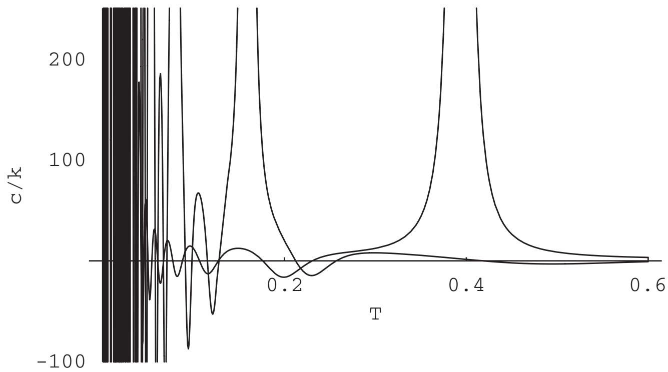

Fig. 3. The case $\mu = 0$. The specific heat for $\gamma = 1$ and $\Omega = 2i$ and $0.8i$.

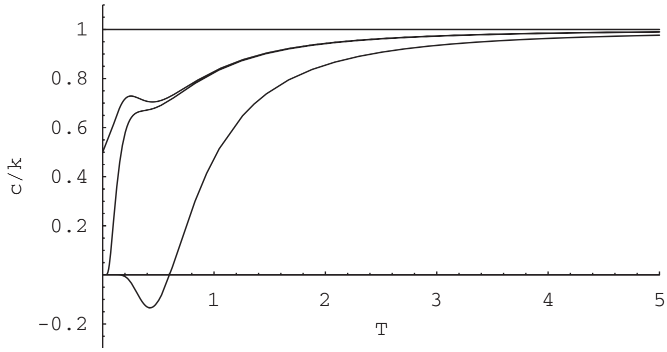

Fig. 4. The case $\mu = 0$. The specific heat for $\gamma = i$ and $\Omega = 0$, 0.2 and 2.

The partition function has well defined limits for $\gamma \to 0$, no friction or for $\Omega \to 0$, critical dumping. We find, respectively,

$$Z(b) = \frac{1}{2\sinh(\hbar\widehat{\omega}b/2)}, \quad Z(b) = \frac{1}{\sqrt{2}\sqrt{1 - \cos(\hbar\gamma b) - \hbar\gamma b \sin(\hbar\gamma b)}}, \tag{64}$$

which are actually the same with the previous ones but now these are one-dimensional. The free energy and also the specific heat of the system with this last partition function is half of that of the section (5), so it has the same points of singularities.

7. Conclusion

We started from a two-dimensional non-commuting space ($[q_1, q_2] = i\theta$) and two independent systems of harmonic oscillators with exponentially varying masses with friction parameter γ. The system is in a magnetic field (squeezed states) with intensity λ. With a linear transformation the problem reduces to that of two other coupled systems in a commuting space ($[Q_1, Q_2] = 0$). The phase space is a standard quantum phase space but that of the second dimension satisfies a deformed commutation relation ($[P_2, Q_2] = i\hbar\mu$). So for $\mu = 0$ we are dealing, in some sense, with a "classical" regime.

We have found the exact propagator of the system which is the main goal of this paper. We first expand the time evolution operator in some kind of normal ordering. With the help of this formula the propagator can be calculated easily and straightforward. We have found as well the Boltzmann statistical density matrix and the partition function, which possesses a denumerable number of singularities as the temperature tends to zero and some zeros. For $\gamma = 0$ the system is equivalent with that of a two independent harmonic oscillator with two deformed frequencies $\widehat{\omega}_1$ and $\widehat{\omega}_2$.

We have found the partition function of the system in the limiting case of critical dumping where the frequencies become zero $\Omega_1 = 0$ and $\Omega_2 = 0$. The specific heat of this system possesses some interesting singularities which disappear in the classical limit $\hbar \to 0$.

For the value of the magnetic intensity $\lambda = \hbar^2/\theta$, the phase space of the second dimension becomes classical ($[P_2, Q_2] = 0$). The final partition function is equivalent with that of an one-dimensional system. For $\gamma = 0$ this system is equivalent with a one-dimensional harmonic oscillator with deformed frequency $\widehat{\omega}$.

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