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Deformed damped harmonic oscillator

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Abstract

In the present paper we study the deformed harmonic oscillator in the presence of friction. We use the following time dependent Hamiltonian:

$$\mathcal{H} = e^{2\gamma t}\frac{a}{2m}\left(\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2\right)^2 + e^{-2\gamma t}\frac{bm\omega^2}{2}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right)^2,$$

where λ, θ are real parameters. Hamiltonian of this type have been used to study dissipation in quantum mechanics.

We find the exact propagator of the system. We find as well the time evolution of the coordinates and momenta operators. Finally, we investigate the thermodynamic properties of the system in Boltzmann statistics. We find the statistical density matrix and the specific heat.

1. Introduction

In the recent few years there is an increasing interesting in the noncommutative geometry [1] for the study of many physical problems. It becomes clear that there is a strong connection of these ideas with string theories and field theories [2] with many applications in solid state and particles physics. Really, the noncommutative geometry arises very naturally from Matrix theory.

Noncommutativity is a mathematical concept expressing the uncertainty. The phase space of ordinary quantum mechanics is a well-known example of noncommuting space. The momenta of a system in the presence of a magnetic field are noncommuting operators as well.

We will study the problem of noncommuting space in two dimensions and we postulate the following commutation relations:

$$[\hat{q}_1, \hat{q}_2] = i\theta, \quad [\hat{p}_1, \hat{p}_2] = i\lambda, \quad [\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}. \tag{1}$$

The first commutation relation among coordinates, expresses the noncommutativity of space and the θ parameter has dimension of $(\text{length})^2$.

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The commutation relation of the momenta operators expresses the existence of a magnetic field. For instance, a homogenous magnetic field along the z-direction, $\vec{H} = (0, 0, H)$ gives the relation

$$[\hat{p}_1, \hat{p}_2] = \mathrm{i}\lambda, \quad \lambda = \hbar^2 \frac{eH}{\hbar c}. \tag{2}$$

We can insert the electromagnetic field naturally with the deformation of the free Hamiltonian. In this paper, we will use the same deformation method to study a system with friction in noncommutative space in two dimensions.

The theory of a system with friction terms has been developed by many authors [3]. If the friction is a linear function of the velocity with friction constant γ the Hamiltonian operator takes the form

$$\mathscr{H} = \mathrm{e}^{2\gamma t} \frac{a}{2m} \hat{p}^2 + \mathrm{e}^{-2\gamma t} V(\hat{q}). \tag{3}$$

Some special potentials have been studied by several authors [4] especially the case of the harmonic oscillator. The above Hamiltonian is not the energy or any constant of motion, but it is rather the generator of the motion. The propagator of some systems with friction terms in electromagnetic fields have been calculated and the Boltzmann statistics of such systems has been studied in Ref. [5].

In this paper we consider the following deformed simple harmonic oscillator [6]:

$$\mathscr{H} = \mathrm{e}^{2\gamma t} \frac{a}{2m} \left(\hat{p}_1 + \frac{\lambda}{2\hbar} \hat{q}_2 \right)^2 + \mathrm{e}^{-2\gamma t} \frac{bm\omega^2}{2} \left(\hat{q}_1 - \frac{\theta}{2\hbar} \hat{p}_2 \right)^2, \tag{4}$$

where γ is the friction constant.

The coordinates and momenta satisfies the usual commutation relations

$$[\hat{q}_1, \hat{q}_2] = [\hat{p}_1, \hat{p}_2] = 0, \quad [\hat{p}_1, \hat{q}_1] = [\hat{p}_2, \hat{q}_2] = -\mathrm{i}\hbar. \tag{5}$$

The Hamiltonian (4) is the deformed Caldirola–Kanai Hamiltonian of the damped harmonic oscillator. Hamiltonian of this type have been used to study dissipation in quantum mechanics [7].

The present paper is organized as follows. In the next section we will find the propagator of the system. We will first expand the time evolution operators in a appropriate normal, so that the propagator can be calculated easily by the action of this operator on delta functions. The propagator becomes a product of two independent distributions for $\mu = 2$ and $\gamma = 0$. Some other critical values of the μ parameter are found in the next section. Next we will evaluate the time evolution of the basic operators $\hat{p}_j$, $\hat{q}_j$ and finally we will find the partition function and the specific heat of the Boltzmann statistics.

2. The propagator of the system

In this section we shall calculate the propagator of the above Hamiltonian. We shall use the formula

$$G(q_1, q_1', q_2, q_2', t) = U(t)\delta(q_1 - q_1')\delta(q_2 - q_2') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k_1 \, \mathrm{d}k_2 \, U(t) \mathrm{e}^{\mathrm{i}k_1(q_1 - q_1')} \mathrm{e}^{\mathrm{i}k_2(q_2 - q_2')}.$$

We expand the time evolution operator in canonical order of the form

$$\begin{aligned} U(t) &= \mathrm{e}^{f_{44}\hat{q}_2^2} \, \mathrm{e}^{(f_{43}/2)(\hat{q}_2\hat{p}_2 + \hat{p}_2\hat{q}_2)} \, \mathrm{e}^{f_{42}\hat{q}_2\hat{q}_1} \mathrm{e}^{f_{41}\hat{q}_2\hat{p}_1} \, \mathrm{e}^{f_{33}\hat{p}_2^2} \, \mathrm{e}^{f_{32}\hat{p}_2\hat{q}_1} \mathrm{e}^{f_{31}\hat{p}_2\hat{p}_1} \, \mathrm{e}^{f_{22}\hat{q}_1^2} \, \mathrm{e}^{(f_{21}/2)(\hat{q}_1\hat{p}_1 + \hat{p}_1\hat{q}_1)} \, \mathrm{e}^{f_{11}\hat{p}_1^2} \\ &= U_2(\hat{p}_2, \hat{q}_2, \hat{p}_1, \hat{q}_1, t) U_1(\hat{p}_1, \hat{q}_1, t) \end{aligned} \tag{6}$$

so the propagator becomes

$$G(q_1, q_1', q_2, q_2', t) = U_2 \delta(q_2 - q_2') U_1 \delta(q_1 - q_1') = U_2 \delta(q_2 - q_2') G_1(q_1, q_1', t). \tag{7}$$

The last function, that is $G_1(q_1, q_1', t)$, is the propagator of the system in the first dimension.

From the equation of motion of the time evolution operator, we find

$$HU(t) = \mathrm{i}\hbar \frac{\partial}{\partial t} U(t) \Rightarrow H = \left(\mathrm{i}\hbar \frac{\partial}{\partial t} U(t) \right) U(t)^{-1}. \tag{8}$$

We perform the above differentiation using the expansion (Eq. (6)) of the operator $U(t)$ and we find

$$\left(\frac{\partial}{\partial t}U\right)U^{-1} = \dot{f}_{44}\hat{q}_2^2 + e^{f_{44}\hat{q}_2^2}((\dot{f}_{43}/2)(\hat{q}_2\hat{p}_2 + \hat{p}_2\hat{q}_2))e^{-f_{44}\hat{q}_2^2} + e^{f_{44}\hat{q}_2^2}e^{(f_{43}/2)(\hat{q}_2\hat{p}_2+\hat{p}_2\hat{q}_2)}(\dot{f}_{42}\hat{q}_2\hat{q}_1)e^{(-f_{43}/2)(\hat{q}_2\hat{p}_2+\hat{p}_2\hat{q}_2)}e^{-f_{44}\hat{q}_2^2} + \cdots. \tag{9}$$

With the help of the operator relation

$$e^{-bA}Be^{bA} = B - b[A,B] + \frac{b^2}{2!}[A,[A,B]] + \cdots$$

and after a simple calculation we can write relation (9) as a polynomial of second order of coordinate and momenta operators. We equate this relation with the Hamiltonian which is a same order polynomial and finally we find a differential system of 10 equations with the 10 unknown functions $f_{ij}(t)$ [8].

The system is rather lengthy but it can be treated by computer programs. We write only the solution.

We make the substitution

$$\mu = 1 + \frac{\theta\lambda}{4\hbar^2}. \tag{10}$$

The angle of the trigonometric functions is

$$\Omega = \sqrt{ab\,\omega^2\mu^2 - \gamma^2}, \tag{11}$$

which becomes zero for the value $\gamma = \pm\sqrt{ab}\,\omega\mu$.

We will use the following time dependent functions:

$$a_1 = e^{\gamma t}\left(\cos(\Omega t) - \frac{\gamma}{\Omega}\sin(\Omega\,cut)\right), \quad b_1 = e^{\gamma t}\frac{\sin(\Omega t)}{\Omega},$$

$$a_2 = e^{-\gamma t}\left(\cos(\Omega t) + \frac{\gamma}{\Omega}\sin(\Omega\,cut)\right), \quad b_2 = e^{-\gamma t}\frac{\sin(\Omega t)}{\Omega}. \tag{12}$$

In the case where the friction parameter does not satisfy the inequalities

$$-\sqrt{ab}\omega\mu < \gamma < \sqrt{ab}\omega\mu \tag{13}$$

then the above trigonometric functions become hyperbolic.

The solution of the system is as follows:

$$f_{44} = -\frac{ia\lambda^2}{8d_1^2\hbar^3 m}a_2 b_1, \quad f_{43} = -\frac{i}{\hbar}\log(d_1)$$
$$f_{42} = \frac{i\lambda}{2\hbar^2\mu}(-1 + a_2), \quad f_{41} = -\frac{ia\lambda}{2\hbar^2 m}b_1,$$
$$f_{33} = \frac{ibb_2 m\omega^2\theta^2}{8\hbar^3\mu^2 d_1^2}(-2 + a_1 + a_2(1-\mu^2)), \quad f_{32} = \frac{ibb_2 m\omega^2\theta}{2\hbar^2 d_1},$$
$$f_{31} = -\frac{i\theta}{2d_1\hbar^2}(d_1 - d_2), \quad f_{22} = -\frac{ibb_2 m\omega^2}{2\hbar d_2},$$
$$f_{21} = -\frac{i}{\hbar}\log\left(\frac{d_2}{d_1}\right), \quad f_{11} = -\frac{ia}{2d_2\hbar m}b_1, \tag{14}$$

where

$$d_1 = \frac{1}{\mu}(1 + a_2(-1+\mu)),$$

$$d_2 = \frac{1}{\mu^2}(a_1 + (2 + a_2(-1+\mu))(-1+\mu)). \tag{15}$$

The ordering of the evolution operator in canonical form, makes it easy, to find the action of the time evolution operator on the coordinates and momentum operators and consequently on the delta functions. This is the main advantage of this method because we can follow step by step the deformation of the system.

In addition, in the next paragraph we will study some interesting limiting cases, which are not clear from the final result.

We proceed the calculations and we find

$$U(t)e^{ik_1(q_1-q_1')}e^{ik_2(q_2-q_2')}=\sqrt{\frac{1}{d_2}}e^{-is_1k_1^2-is_2k_2^2-is_3k_1k_2+ik_1u_1+ik_2u_2+iu_3}. \tag{16}$$

The functions s_1, s_2 plays the main role for the integration with respect to k_1 and k_2. If one of these parameters is zero the integration leaves a corresponding delta function to the result. These are functions of f_{ij} and have the form

$$s_1=\hbar^2 f_{11},\quad s_2=\hbar^2(f_{33}-\hbar^2 f_{22}f_{31}^2),\quad s_3=\hbar^2 e^{-i\hbar f_{43}}f_{31}. \tag{17}$$

Under certain conditions the integration of formula (16) gives the propagator of the system

$$G(q_1,q_1',q_2,q_2',t)=\sqrt{\frac{1}{d_2(4s_1s_2-s_3^2)}}\exp\left\{\frac{i}{4s_1s_2-s_3^2}(s_1u_1^2+s_2u_2^2-s_3u_1u_2)+iu_3\right\}. \tag{18}$$

We substitute the functions f_{ij} and we finally find

$$s_1=\frac{ab_1\hbar}{2d_2m},\quad s_2=\frac{bb_2m\omega^2\theta^2}{8\hbar d_2},\quad s_3=\frac{\theta}{2d_2}(d_1-d_2). \tag{19}$$

The other factors of the formula (18) are

$$u_3=-\frac{\lambda}{2\hbar^2 d_2\mu^2}(-2+\mu+a_1+a_2(1-\mu))q_1q_2-\frac{bm\omega^2b_2}{2\hbar d_2}q_1^2-\frac{ab_1\lambda^2}{8\hbar^3md_2}q_2^2,$$

$$u_1=\frac{d_1}{d_2}q_1-\frac{ab_1\lambda}{2d_2\hbar m}q_2-q_1',\quad u_2=\frac{d_1}{d_2}q_2+\frac{bb_2m\omega^2\theta}{2d_2\hbar}q_1-q_2'. \tag{20}$$

So finally the propagator becomes

$$\begin{aligned}G(q_1,q_1',q_2,q_2',t)=\frac{\mu}{\theta}\sqrt{\frac{2}{d_0}}\exp\left\{\frac{i}{2\hbar d_0}\left(\frac{2\hbar}{\theta}(q_2-q_2')(q_1d_3+q_1'd_4)-\frac{\lambda}{2\hbar}(q_1-q_1')(q_2d_4+q_2'd_3)\right)\right\}\\ \times\exp\left\{\frac{i}{2\hbar}\frac{\mu^2}{d_0}\left(\frac{4\hbar^2}{\theta^2}b_1\frac{a}{2m}(q_2-q_2')^2+b_2\frac{b}{2}m\omega^2(q_1-q_1')^2\right)\right\},\end{aligned} \tag{21}$$

where

$$d_3=1-a_2,\quad d_4=1-a_1,$$

$$d_0=\frac{1}{2}(d_3+d_4)=1-\cosh(\gamma t)\cos(\Omega t)+\frac{\gamma}{\Omega}\sinh(\gamma t)\sin(\Omega t). \tag{22}$$

The propagator is a product of two exponentials. The second exponential is a distribution of two systems with respect to the two dimensions q_1 and q_2.

The first exponential expresses the coupling of the two systems. It depends on the fractions d_3/d_0 and d_4/d_0 which contain the parameters of the Hamiltonian and the friction parameter γ. In the case of small values of the friction γ that is $\gamma^2<ab\omega^2\mu^2$ these terms have an infinite number of singularities. For the limit $\gamma\to 0$ both these fractions reduce to one and so the coupling terms of the systems becomes independent of the Hamiltonian.

In the limit $t \to 0$ the propagator does not tend to the delta's function but

$$\lim_{t\to 0} G(q_1, q_1', q_2, q_2', t) = \exp\left\{\frac{\mathrm{i}}{2\hbar}\left(\frac{2\hbar}{\theta}(q_1+q_1')(q_2-q_2') - \frac{\lambda}{2\hbar}(q_1-q_1')(q_2+q_2')\right)\right\}$$
$$\times\delta(q_1-q_1')\delta(q_2-q_2') = \delta(q_1-q_1')\delta(q_2-q_2'). \quad (23)$$

The additional term does not depend on the Hamiltonian or the friction parameter γ. This indicates that the deformation made is not artificial.

The effect of the parameter θ on the final result is to substitute a Dirac delta function $\delta(q_2 - q_2')$ by a Gaussian function. This is obvious if we write the propagator in the following equivalent form.

$$G(q_1, q_1', q_2, q_2', t)$$
$$= \frac{\mu}{\theta}\sqrt{\frac{2}{d_0}}\exp\left\{\frac{\mathrm{i}}{4\hbar d_0}\left(\frac{m}{a\mu^2 b_1}(q_1 d_3 + q_1' d_4)^2 + \frac{\lambda^2}{4\hbar^2}\frac{1}{bm\omega^2\mu^2 b_2}(q_2 d_4 + q_2' d_3)^2\right)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{\mu^2 b_2}{d_0}\frac{b}{2}m\omega^2\left(q_1 - q_1' - \frac{\lambda}{2\hbar}\frac{1}{bm\omega^2\mu^2 b_2}(q_2 d_4 + q_2' d_3)\right)^2\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{\mu^2 b_1}{d_0}\frac{a}{2m}\frac{4\hbar^2}{\theta^2}\left(q_2 - q_2' + \frac{\theta}{2\hbar}\frac{m}{a\mu^2 b_1}(q_1 d_3 + q_1' d_4)\right)^2\right\}. \quad (24)$$

So the propagator depends on the following parameters

$$\theta \quad \text{and} \quad B = \frac{\lambda}{\hbar^2}. \quad (25)$$

In the case where there is a magnetic field then the length

$$L = \sqrt{B} = \frac{eH}{\hbar c} \quad (26)$$

is mainly used to experimental applications for the cyclotron resonances.

The value $\sqrt{\theta}$ is the minimum length attainable in the space of the dimension q_2. The best localization in this space is within a cell of area θ [9].

The same problem appears for the limit $\lambda \to \infty$. This is due to the fact that this parameter is connecting with a constant magnetic field of the form

$$\vec{H} = (0, 0, H) \to \lambda = \frac{e}{c}H.$$

On the contrary the limits $\lambda \to 0$ and $\theta \to \infty$ are well defined. Notice that

$$\lim_{\theta\to\infty}\frac{\mu}{\theta} = \lim_{\theta\to\infty}\left(\frac{1}{\theta} + \frac{\lambda}{4\hbar^2}\right) = \frac{\lambda}{4\hbar^2}.$$

This is not strange because, for instant, the limit $\lambda \to 0$ is equivalent to the limit $c \to \infty$.

In the sequel we will evaluate the propagator for the limiting case where

$$\Omega = 0, \quad \gamma = \pm\sqrt{ab}\omega\mu, \quad \omega = \pm\frac{\gamma}{\sqrt{ab}\,\mu}. \quad (27)$$

We take the limit to the final result and we find

$$G(q_1, q_1', q_2, q_2', t) = \frac{\mu}{\theta}\sqrt{\frac{2}{d_0}}\exp\left\{\frac{\mathrm{i}}{2\hbar d_0}\left(\frac{2\hbar}{\theta}(q_2 - q_2')(q_1' d_3 + q_1 d_4) - \frac{\lambda}{2\hbar}(q_1 - q_1')(q_2 d_3 + q_2' d_4)\right)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}bm\omega^2 \mathrm{e}^{-\gamma t} t\mu^2}{4\hbar d_0}(q_1 - q_1')^2 + \frac{4\hbar^2}{\theta^2}\frac{\mathrm{i}a\,\mathrm{e}^{\gamma t} t\mu^2}{4\hbar m d_0}(q_2 - q_2')^2\right\}, \quad (28)$$

where

$$d_3 = 1 - e^{\gamma t}(1 - \gamma t), \quad d_4 = 1 - e^{-\gamma t}(1 + \gamma t),$$

$$d_0 = \tfrac{1}{2}(d_3 + d_4) = 1 - \cosh(\gamma t) + \gamma t \sinh(\gamma t). \tag{29}$$

3. Some special values of the parameters

From the functions f_{ij}, Eq. (14) we can see that we have the following critical values of the μ parameter: For

$$\mu = 1 \Rightarrow \lambda\theta = 0. \tag{30}$$

If $\lambda \to 0$ the functions with the subscript 4 that is f_{44}, f_{43}, f_{42} and f_{41} vanish. But this is not so dramatic because these are coefficient of first-order derivative. In addition the s_j (Eq. (18)) do not change drastically. So we can find this limit from the final result.

If $\theta \to 0$, the functions with the subscript 3, that is f_{33}, f_{31} and f_{32} vanish. The first two functions contains second-order derivatives that is

$$f_{33}\hat{p}_2^2 = -\hbar^2 f_{33}\frac{\partial^2}{\partial q_2^2}, \quad f_{31}\hat{p}_2\hat{p}_1 = -\hbar^2 f_{31}\frac{\partial^2}{\partial q_2 \partial q_1}. \tag{31}$$

These factors contribute essential terms in the final result. In addition, the function s_2 and s_3 vanish so the integration with respect to k_2 leaves the $\delta(q_2 - q_2')$ in the final result.

In the physical content, θ is not a typical parameter which we can increase or decrease continuously. It is more natural to think of θ as the deformation of ordinary coordinates. The coordinates and the additional law are the same but now we have a different multiplication law, which is often denoted by $q_1 \star q_2$, star product. This parameter expresses the noncommutativity of space and decreasing $\theta \to 0$ means that we go from noncommutativity to commutativity. So this limit has to be taken before integration. This limit leads from a Gaussian distribution to a delta function.

We execute only the integration with respect of k_1 of function (16) and we find the propagator

$$G(q_1, q_1', q_2, q_2', t) = \sqrt{\frac{2m\Omega}{\mathrm{i}a\hbar b1}}\exp\left\{-\frac{\mathrm{i}}{\hbar}\frac{\lambda}{2\hbar}q_2(q_1 - q_1') + \frac{\mathrm{i}m\Omega}{2a\hbar b_1}(a_2 q_1^2 + a_1 {q_1'}^2 - 2q_1 q_1')\right\}\delta(q_2 - q_2'), \tag{32}$$

where now we have $\mu = 1$ and $\Omega = \sqrt{ab\,\omega^2 - \gamma^2}$.

This formula is the product of an harmonic oscillator, in the first dimension and a delta function in the second dimension. The third term depend on λ and vanish if $\lambda = 0$.

Another interesting case is the value

$$\mu = 2 \to \frac{2\hbar}{\theta} = \frac{\lambda}{2\hbar}. \tag{33}$$

In this case and for $\gamma = 0$ where there is no friction terms, the function f_{31} vanish. In addition, the term s_3 which is the coefficient of $k_1 k_2$ and the coefficient of the product $q_1 q_2$ in the u_3 vanish as well.

We find

$$G(q_1, q_1', q_2, q_2', t) = \frac{\mu}{\theta}\sqrt{\frac{2}{d_0}}\exp\left\{\frac{\mathrm{i}}{2\hbar d_0}\frac{2\hbar}{\theta}(2q_2 q_1' d_4 - 2q_1 q_2' d_3 + (q_1 q_2 + q_1' q_2')(d_3 - d_4))\right\}$$
$$\times \exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{\mu^2}{d_0}\left(\frac{4\hbar^2}{\theta^2}b_1\frac{a}{2m}(q_2 - q_2')^2 + b_2\frac{b}{2}m\omega^2(q_1 - q_1')^2\right)\right\}, \tag{34}$$

where now we have $\mu = 2$ and $\Omega = \sqrt{4ab\,\omega^2 - \gamma^2}$.

The factor

$$d_3 - d_4 = a_1 - a_2 = 2\left(\sinh(\gamma t)\cos(\Omega t) - \frac{\gamma}{\Omega}\cosh(\gamma t)\sin(\Omega t)\right) \tag{35}$$

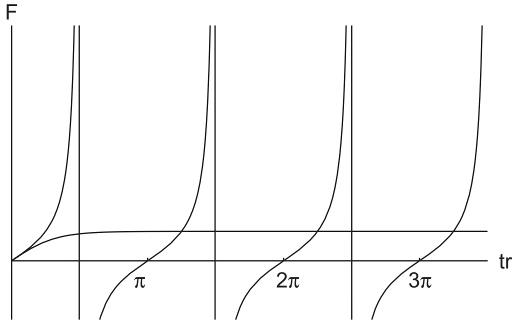


Fig. 1. The solutions of Eq. (36) for $\gamma = \sqrt{ab}\omega\mu/\sqrt{2}$.

of the above formula for $\gamma = 0$ vanish and we have two independent propagators. In the case of strong friction that is $\gamma^2 > ab\omega^2\mu^2$ this factor vanish only for $t = 0$. On the contrary for small friction $\gamma^2 < ab\omega^2\mu^2$ there are some other values of time different of zero, the solutions of the following equation, which also vanish this factor.

$$F = \tanh(\gamma t) = \frac{\gamma}{\Omega}\tan(\Omega t). \tag{36}$$

The solutions of this Eq. (36) are the intersections of the functions $F = \tanh(\gamma t)$ and $F = (\gamma/\Omega)\tan(\Omega t)$ (Fig. 1).

The resulting propagator (34) contains the factors $1/\theta$ and $1/\hbar$. It is interesting to note that in this formula we can substitute instead of $1/\theta$ the term $\lambda/4\hbar^2$. We can as well eliminate the hole dependence of the formula on the $\hbar$ ($\hbar = \sqrt{\lambda\theta}/2$) and so the propagator has no any $\hbar$ at all. It is clear that we have three parameters and one relation connecting them, so we can eliminate one parameter from the final result.

The last limiting case comes from the function f_{33} of Eq. (14). This is

$$\mu = \pm\sqrt{2} \;\Rightarrow\; \frac{\lambda}{2\hbar} = \frac{2\hbar}{\theta}(-1 \pm \sqrt{2}), \quad \Omega = \sqrt{2ab\,\omega^2 - \gamma^2}, \tag{37}$$

where the function f_{33} becomes

$$\begin{aligned} f_{33} &= \frac{\mathrm{i}bb_2 m\omega^2\theta^2}{8\hbar^3\mu^2 d_1^2}(-2 + a_1 - a_2) \\ &= \frac{\mathrm{i}bb_2 m\omega^2\theta^2}{4\hbar^3\mu^2 d_1^2}\left(-1 + \cos(\Omega t)\sinh(\gamma t) - \frac{\gamma}{\Omega}\cosh(\gamma t)\sin(\Omega t)\right). \end{aligned} \tag{38}$$

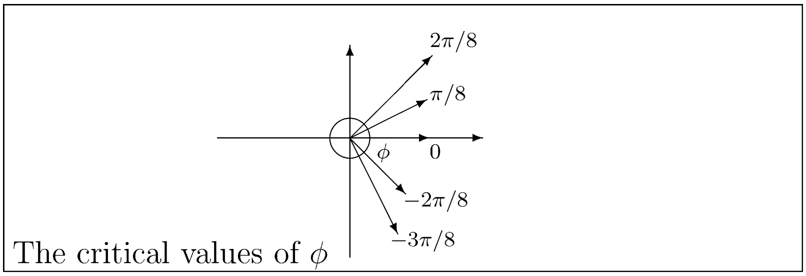
The term $a_1 - a_2$ vanish for $\gamma = 0$, while for $\gamma \neq 0$ this term vanish for some values of the time t.

We define the angle ϕ from the relation

$$\tan\phi = \frac{\lambda/2\hbar}{2\hbar/\theta} = \frac{\lambda\theta}{4\hbar^2}.$$

We write all the above critical values of μ and the corresponding values of the angle ϕ.

$$\begin{aligned} &\mu \to -\sqrt{2}, \quad 0, \quad 1, \quad +\sqrt{2}, \quad 2, \\ &\frac{\lambda\theta}{4\hbar^2} \to -\sqrt{2} - 1, \quad -1, \quad 0, \quad \sqrt{2} - 1, \quad 1, \\ &\phi \to -3\frac{\pi}{8}, \quad -2\frac{\pi}{8}, \quad 0, \quad \frac{\pi}{8}, \quad 2\frac{\pi}{8}. \end{aligned} \tag{39}$$



From the above values we do not simply throw away the negative values of the product $(\lambda/2\hbar)(\theta/2\hbar)$. We assume first that the θ parameter is positive otherwise we can interchange the coordinates q_1, q_2. On the other hand, the parameter λ is connected with the momenta. So in order to keep the negative solutions we can assume that these states correspond to some kind of negative—effective—mass. But this assumption of course needs further study. That is

$$\theta > 0, \quad \lambda > 0, \quad m > 0, \quad \theta > 0, \quad \lambda < 0, \quad m < 0. \tag{40}$$

As far as I know negative effective mass, has been pointed out at first by Dousmanis [10].

We forget for instant the problem of negative μ and we consider the propagator (21) as a function of the parameter μ. We denote as $G(\mu)$ and $G(-\mu)$ the propagators with positive and negative μ. After a little algebra we can prove the following relation:

$$G(-\mu) = -G(\mu)\exp\left\{\frac{\mathrm{i}\mu}{2\hbar d_0}\left(\frac{2\hbar}{\theta}(q_1 - q_1')(d_4 q_2 + d_3 q_2') - \frac{\lambda}{2\hbar}\frac{1}{\mu^2 - 1}(q_2 - q_2')(d_3 q_1 + d_4 q_1')\right)\right\}. \tag{41}$$

So the value $\mu = \pm\sqrt{2}$ implies that $\mu^2 - 1 = 1$ and the above relation takes the following symmetric form:

$$G(-\sqrt{2}) = -G(\sqrt{2})\exp\left\{\frac{\mathrm{i}\sqrt{2}}{2\hbar d_0}\left(\frac{2\hbar}{\theta}(q_1 - q_1')(d_4 q_2 + d_3 q_2') - \frac{\lambda}{2\hbar}(q_2 - q_2')(d_3 q_1 + d_4 q_1')\right)\right\}. \tag{42}$$

4. Time evolution of coordinates and momenta

We can now find the time evolution of the momenta and position operators. We define the vector

$$\hat{P}(t) = (\hat{p}_1(t), \quad \hat{q}_1(t), \quad \hat{p}_2(t), \quad \hat{q}_2(t)). \tag{43}$$

The expansion of the time evolution operator in the normal ordered form (6) can be used to find easily the time evolution of the momentum and position operators. The time evolution of the operators is given by the following linear equation:

$$\hat{P}(t) = U(t)\hat{P}(0)U(t)^{-1} = \hat{P}(0)X(t). \tag{44}$$

The transformation matrix has now the form

$$X(t) = \frac{1}{\mu}\left\{\begin{matrix} \mu - 1 & 0 & 0 & -\frac{\lambda}{2\hbar} \\ 0 & \mu - 1 & \frac{\theta}{2\hbar} & 0 \\ 0 & \frac{\lambda}{2\hbar} & 1 & 0 \\ -\frac{\theta}{2\hbar} & 0 & 0 & 1 \end{matrix}\right\} + \frac{1}{\mu}\begin{pmatrix} 1 & 0 & 0 & \frac{\lambda}{2\hbar} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\theta}{2\hbar} & 0 & 0 & \mu - 1 \end{pmatrix} \mathrm{e}^{\gamma t}\left(\cos(\Omega t) - \frac{\gamma}{\Omega}\sin(\Omega t)\right)$$

$$+\frac{1}{\mu}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -\frac{\theta}{2\hbar} & 0\\ 0 & -\frac{\lambda}{2\hbar} & \mu-1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\mathrm{e}^{-\gamma t}\Big(\cos(\Omega t)+\frac{\gamma}{\Omega}\sin(\Omega t)\Big)$$

$$+\begin{pmatrix}0 & 1 & -\frac{\theta}{2\hbar} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \frac{\theta}{2\hbar} & -\frac{\theta^2}{4\hbar^2} & 0\end{pmatrix}bm\omega^2\frac{\mathrm{e}^{-\gamma t}\sin(\Omega t)}{\Omega}+\begin{pmatrix}0 & 0 & 0 & 0\\ -1 & 0 & 0 & -\frac{\lambda}{2\hbar}\\ \frac{\lambda}{2\hbar} & 0 & 0 & \frac{\lambda^2}{4\hbar^2}\\ 0 & 0 & 0 & 0\end{pmatrix}\frac{a}{m}\frac{\mathrm{e}^{\gamma t}\sin(\Omega t)}{\Omega}. \tag{45}$$

This equation coincide with the "classical" equation of motion.

The operators $\hat{P}(t)$ preserve the basic commutation relations, namely

$$[\hat{p}_j(t),\hat{q}_j(t)]=[\hat{p}_j(0),\hat{q}_j(0)]=\mathrm{i}\hbar,\quad j=1,2,$$

$$[\hat{p}_1(t),\hat{q}_2(t)]=[\hat{p}_1(0),\hat{q}_2(0)]=0,\quad [\hat{p}_2(t),\hat{q}_1(t)]=[\hat{p}_2(0),\hat{q}_1(0)]=0. \tag{46}$$

The operators

$$\hat{p}_1+\frac{\lambda}{2\hbar}\hat{q}_2,\quad \hat{q}_1-\frac{\theta}{2\hbar}\hat{p}_2 \tag{47}$$

satisfy the equation

$$\begin{pmatrix}p_1(t)+\frac{\lambda}{2\hbar}q_2(t)\\ q_1(t)-\frac{\theta}{2\hbar}p_2(t)\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{-\gamma t}(\cos(\Omega t)+\frac{\gamma}{\Omega}\sin(\Omega t)) & bm\omega^2\mu\mathrm{e}^{-\gamma t}\frac{\sin(\Omega t)}{\Omega}\\ -\frac{a}{m}\mu\,\mathrm{e}^{\gamma t}\frac{\sin(\Omega t)}{\Omega} & \mathrm{e}^{\gamma t}(\cos(\Omega t)-\frac{\gamma}{\Omega}\sin(\Omega t))\end{pmatrix}\begin{pmatrix}p_1+\frac{\lambda}{2\hbar}q_2\\ q_1-\frac{\theta}{2\hbar}p_2\end{pmatrix}. \tag{48}$$

This transformation is a normal symmetric rotation, while for the transformation (44) this symmetry is broken.

5. Boltzmann statistics of quantum friction

As is well known [11] with the substitution

$$t\to -\mathrm{i}\hbar\beta,\quad \beta=\frac{1}{kT}$$

we obtain the density matrix $\rho(\vec{r},\vec{r},\beta)$ of the system. From propagator (21) we find

$$\rho(\vec{r},\vec{r}',\beta)=\frac{\mu}{\theta}\sqrt{\frac{2}{d_0}}\exp\left\{\frac{\mathrm{i}}{2\hbar d_0}\left(\frac{2\hbar}{\theta}(q_2-q_2')(q_1d_3+q_1'd_4)-\frac{\lambda}{2\hbar}(q_1-q_1')(q_2d_4+q_2'd_3)\right)\right\}$$
$$\times\exp\left\{\frac{\mathrm{i}}{2\hbar}\frac{\mu^2}{d_0}\left(\frac{4\hbar^2}{\theta^2}b_1\frac{a}{2m}(q_2-q_2')^2+b_2\frac{b}{2}m\omega^2(q_1-q_1')^2\right)\right\}. \tag{49}$$

It has the same form but now the functions a_j and b_j are as follows:

$$a_1=\mathrm{e}^{-\mathrm{i}\gamma\hbar\beta}\Big(\cosh(\Omega\hbar\beta)+\mathrm{i}\frac{\gamma}{\Omega}\sinh(\Omega\hbar\beta)\Big),\quad b_1=-\mathrm{i}\mathrm{e}^{-\mathrm{i}\gamma\hbar\beta}\frac{\sinh(\Omega\hbar\beta)}{\Omega}, \tag{50}$$

$$a_2=\mathrm{e}^{\mathrm{i}\gamma\hbar\beta}\Big(\cosh(\Omega\hbar\beta)-\mathrm{i}\frac{\gamma}{\Omega}\sinh(\Omega\hbar\beta)\Big),\quad b_2=-\mathrm{i}\mathrm{e}^{\mathrm{i}\gamma\hbar\beta}\frac{\sinh(\Omega\hbar\beta)}{\Omega}. \tag{51}$$

The special case where $\vec{r}=\vec{r}'$ gives

$$\rho(\vec{r},\vec{r},\beta)=\frac{\mu}{\theta}\sqrt{\frac{2}{d_0}}, \tag{52}$$

where

$$d_0 = 1 - \cos(\gamma\hbar\beta)\cosh(\Omega\hbar\beta) - \frac{\gamma}{\Omega}\sin(\gamma\hbar\beta)\sinh(\Omega\hbar\beta). \tag{53}$$

Finally, executing an appropriate integration we find the partition function:

$$Z(\beta) = \int \rho(\vec{r}, \vec{r}, \beta)\,\mathrm{d}q_1\,\mathrm{d}q_2 = \frac{S\mu}{\theta}\sqrt{\frac{2}{d_0}}. \tag{54}$$

For

$$S = \frac{\theta}{2\mu} \tag{55}$$

the partition function coincide with the known one of the simple damped harmonic oscillator.

The free energy F and the specific heat C of the system is defined from the relations

$$F = -kT\log(Z(\beta)), \quad C = -T\frac{\partial^2 F}{\partial T^2}. \tag{56}$$

So we find

$$F = -KT\log\left(\frac{S\mu}{\theta}\right) + \frac{1}{2}kT\log\left(\frac{d_0}{2}\right), \tag{57}$$

$$\frac{C}{k} = -\frac{ab\omega^2\mu^2\hbar^2\beta^2}{4d_0^2}\left(1 + \cos(2\gamma\hbar\beta) + \frac{\gamma^2}{\Omega^2}(1 - \cosh(2\hbar\Omega\beta)) - 2\cos(\gamma\hbar\beta)\cosh(\hbar\Omega\beta) + 2\frac{\gamma}{\Omega}\sin(\gamma\hbar\beta)\sinh(\hbar\Omega\beta)\right). \tag{58}$$

In the region of low temperature, the specific heat exhibits an infinite number of singularities for weak friction $\gamma < \sqrt{ab}\omega\mu)$ [12]. (Fig. 2). In the case of strong friction $\gamma > \sqrt{ab}\omega\mu)$ there are some values of the temperature that it becomes zero (Fig. 3).

For high temperatures we have

$$\lim_{T\to\infty}\frac{C}{k} = 1. \tag{59}$$

If $\gamma = 0$ then Eq. (58) is reduced to the specific heat of an harmonic oscillator:

$$\frac{C}{k} = \frac{ab\omega^2\mu^2\hbar^2\beta^2}{4\sinh^2(\frac{1}{2}\sqrt{ab}\,\omega\mu\hbar\beta)}. \tag{60}$$

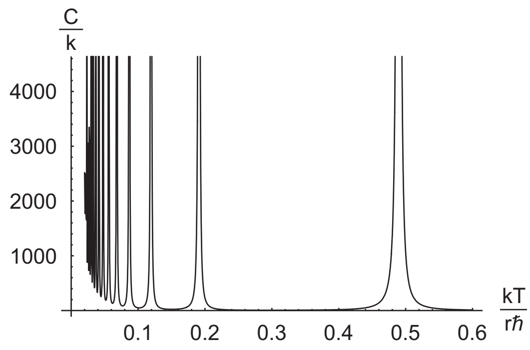


Fig. 2. The specific heat for weak friction $\gamma = \frac{1}{2}\sqrt{ab}\omega\mu$.

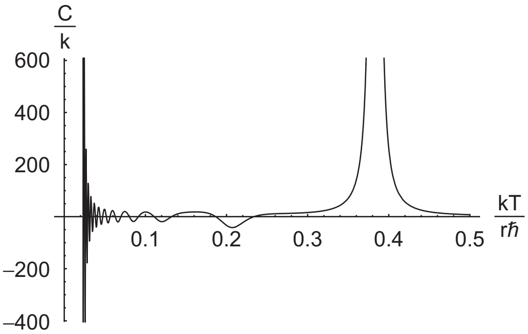


Fig. 3. The specific heat for strong friction $\gamma = 2\sqrt{ab}\omega\mu$.

6. Conclusion

In the present paper, we have examined the deformed harmonic oscillator $\mathscr{H}$ for dissipative systems. We have found the exact propagator of the system. We first expand the time evolution operator in some kinds of normal ordering. With the help of this formula the propagator (Eq. (21)) and the time evolution of the coordinates and momenta (Eq. (44)) can be calculated easily and straightforward.

The propagator is a product of two exponential functions. The second exponential is the propagator of two systems with respect to the two dimensions. The first one expresses the coupling of the two systems. We study also some interesting limiting cases of the parameter μ the most interesting of which are the values $\mu = \pm\sqrt{2}$, which need further study. The value $\mu = 2$ gives two systems where there are no coupling terms when $\gamma = 0$.

We finally find the Boltzmann statistical density matrix, the partition function and the specific heat. For low temperatures we find that the specific heat exhibits an infinite number of singularities and for strong friction some zeros.

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