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HARMONIC OSCILLATOR IN NONCOMMUTING TWO-DIMENSIONAL SPACE

ANTONY STREKLAS

Department of Mathematics, University of Patras, 26500 Patras, Greece
streklas@math.upatras.gr

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In the present paper, we study a two-dimensional harmonic oscillator in a constant magnetic field in noncommuting space. We use the following Hamiltonian

$$\mathcal{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m\omega_1^2\hat{q}_1^2 + \frac{1}{2}m\omega_2^2\hat{q}_2^2\,,$$

with commutation relations $[\hat{q}_1, \hat{q}_2] = i\theta$, $[\hat{p}_1, \hat{p}_2] = i\lambda$ and $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$.

The parameter λ expresses the presence of the magnetic field. We find the exact propagator of the system and the time evolution of the basic operators. We prove that the system is equivalent to a two-dimensional system where the operators of momentum and coordinates of the second dimension satisfy a deformed commutation relation $[\hat{Q}_2, \hat{P}_2] = i\hbar\mu$. The deformation parameter, μ, depends on λ and θ, and is independent of the Hamiltonian. Finally, we investigate the thermodynamic properties of the system in Boltzmann statistics. We find the statistical density matrix and the partition function, which is equivalent to that of a two-dimensional harmonic oscillator with two deformed frequencies Ω_1 and Ω_2.

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1. Introduction

In recent years, there is an increasing interest in the noncommutative geometry[1] for the study of many physical problems. It becomes clear that there is a strong connection of these ideas with string theories and field theories,[2] with many applications in solid-state and particle physics. Really, the noncommutative geometry arises very naturally from Matrix theory.[3]

Noncommutativity is a mathematical concept expressing the uncertainty. The phase space of ordinary quantum mechanics is a well-known example of noncommuting space. The momenta of a system in the presence of a magnetic field are noncommuting operators as well .

We will study the problem of noncommuting space in two-dimensions,[4,5] and we postulate the following commutation relations

$$[\hat{q}_1, \hat{q}_2] = i\theta\,, \quad [\hat{p}_1, \hat{p}_2] = i\lambda\,, \quad [\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}\,. \tag{1}$$

The first commutation relation among coordinates expresses the non commutativity of space, and the θ parameter has dimension of $(\text{length})^2$. So the position of the particle cannot be localized in the space. The commutation relation of the momenta operators expresses the existence of a magnetic field.

2. Canonical Order of the Boson Operators

Any function $f(a, a^+)$ of the boson operators a, and a^+ where

$$[a, a^+] = 1\,, \tag{2}$$

is defined by its power series expansion

$$f(a, a^+) = \sum_{j_1} \cdots \sum_{j_k} a^{+j_1} a^{j_2} \cdots a^{+j_n} a^{j_k} f(j_1, j_2, \ldots, j_k)\,. \tag{3}$$

We can use the commutation relation (2) repeatedly to rearrange the operators a, a^+, in order to write the function $f(a, a^+)$ in a desired form. We say that the function $f(a, a^+)$ is in normal order form or anti-normal order[6] if we can write

$$f^{(n)}(a, a^+) = \sum_{j,k} f^{(n)}_{jk} a^{+j} a^k\,, \quad f^{(a)}(a, a^+) = \sum_{j,k} f^{(a)}_{jk} a^j a^{+k}\,. \tag{4}$$

Since the normal and anti-normal form of an analytic function are unique, we can establish a one to one correspondence between either operator functions $f^{(n)}(a, a^+)$, $f^{(a)}(a, a^+)$ and ordinary functions of a complex variable a through the relations

$$f^{(n)}(a, a^*) = \langle a|f(a, a^+)|a\rangle = \mathrm{Tr}(|a\rangle\langle a|f(a, a^+))\,, \tag{5}$$

$$f^{(a)}(a, a^+) = \int \frac{d^2 a'}{\pi} |a\rangle\langle a| f^{(a)}(a', a'^*)\,, \tag{6}$$

where $|a\rangle$ is the eigenvector of the annihilation operator with eigenvalue a. That is

$$a|a\rangle = a|a\rangle\,, \quad \langle a|a^+ = \langle a|a^*\,, \quad \langle a|a\rangle = 1\,. \tag{7}$$

Some other orderings have been studied in the references, for instance, the Weyl correspondence,[7,8] which leads to the Wigner distribution function.

In this paper, we will find the normal ordered form of the time evolution operator

$$U(t) = e^{-i\frac{t}{\hbar}\mathcal{H}(\hat{\mathbf{p}}, \hat{\mathbf{q}})} \tag{8}$$

where the Hamiltonian $\mathcal{H}(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ is quadratic with respect to its arguments.

We shall write the time evolution operator as a product of exponential operators of the form $\exp\{f_{jk}\hat{q}_j\hat{p}_k\}$, that is

$$U(t) = \prod_{jkmn} e^{f_{jk}(t)\hat{q}_m\hat{q}_n} e^{f_{jk}(t)\hat{q}_m\hat{p}_n} e^{f_{jk}(t)\hat{p}_m\hat{p}_n}\,. \tag{9}$$

To find the unknown functions f_{jk}, we differentiate[9] the above operator with respect to time t and find the operator

$$\left(i\hbar\frac{\partial}{\partial t}U(t)\right) U(t)^{-1} = \mathcal{H}\,. \tag{10}$$

With the help of the operator relation

$$e^{-bA}Be^{bA} = B - b[A,B] + \frac{b^2}{2!}[A,[A,B]] + \cdots \tag{11}$$

we can write the above operator $i\hbar\dot{U}U^{-1}$ as a polynomial of second order of coordinate and momenta operators.

We equate this relation with the Hamiltonian, which is a same-order polynomial, and finally find a first-order differential system of the unknown functions $f_{jk}(t)$.

The unknown functions $f_{jk}(t)$ satisfy the following initial conditions

$$f_{jk}(0) = 0\,. \tag{12}$$

As an example, we find the propagator of the simple harmonic oscillator

$$\mathcal{H}(\hat{p}_1, \hat{q}_1) = k_{11}\hat{p}_1^2 + k_{21}(\hat{p}_1\hat{q}_1 + \hat{q}_1\hat{p}_1) + k_{22}\hat{q}_1^2\,, \tag{13}$$

where

$$[\hat{q}_1, \hat{p}_1] = c_{21} = i\hbar\,. \tag{14}$$

The differential system is as follows

$$\begin{aligned} i\hbar\dot{f}_{22} &= 4c_{21}^2 f_{22}^2 k_{11} - 2c_{21}f_{22}k_{21} + k_{22}\,,\\ i\hbar\dot{f}_{21} &= -4c_{21}f_{22}k_{11} + k_{21}\\ i\hbar\dot{f}_{11} &= e^{-2c_{21}f_{21}}k_{11}\,. \end{aligned} \tag{15}$$

To solve the differential system, we set

$$f_{21} = \frac{1}{c_{21}}\log(x_{11})\,, \quad f_{22} = \frac{1}{2c_{21}}\frac{x_{21}}{x_{11}}\,, \quad f_{11} = -\frac{1}{2c_{21}}\frac{x_{12}}{x_{11}}\,. \tag{16}$$

We can prove that the new unknown functions x_{jk} satisfy the following classical equations of motion

$$\begin{pmatrix} \dot{x}_{11} & \dot{x}_{12} \\ \dot{x}_{21} & \dot{x}_{22} \end{pmatrix} = \begin{pmatrix} k_{21} & -2k_{11} \\ 2k_{22} & -k_{21} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \tag{17}$$

with the following initial conditions

$$x_{jk}(0) = \delta_{jk}\,. \tag{18}$$

The transformation matrix $X(t) = x_{jk}(t)$ gives the time evolution of the basic operators

$$(p_1(t), q_1(t)) = (p_1, q_1)\begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}. \tag{19}$$

Because the commutation relation is conserved, we conclude the following condition

$$\det|X(t)| = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t) = 1\,. \tag{20}$$

Assuming that we have found the functions f_{jk}. We use the following representation of the basic operators

$$\hat{p}_1 \to -c_{21}\partial_{q_1}\,, \quad \hat{q}_1 \to q_1\,, \tag{21}$$

and we find the propagator of the system

$$\begin{aligned} U_1(t)\delta(q_1 - q_1') &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{e^{c_{21}f_{21}}}}\int_{-\infty}^{\infty} dk e^{f_{22}q_1^2} e^{-c_{21}f_{21}q_1\partial_{q_1}} e^{c_{21}^2 f_{11}\partial_{q_1}^2} e^{ik(q_1-q_1')} \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{e^{c_{21}f_{21}}}}\, e^{f_{22}q_1^2}\int_{-\infty}^{\infty} dk e^{-k^2 c_{21}^2 f_{11}} e^{ik(e^{-c_{21}f_{21}}q_1 - q_1')} \Longrightarrow \\ G_1(\tau_1, \tau_1', t) &= \frac{1}{\sqrt{4c_{21}^2 e^{c_{21}f_{21}} f_{11}}}\exp\left\{-\frac{1}{4c_{21}^2 f_{11}}(e^{-c_{21}f_{21}}\tau_1 - \tau_1')^2 + f_{22}\tau_1^2\right\}. \end{aligned} \tag{22}$$

We can write the propagator of the system using the classical solutions x_{jk}. The result is

$$G_1(\tau_1, \tau_1', t) = \frac{1}{\sqrt{-2c_{21}x_{12}}}\exp\left\{-\frac{1}{2c_{21}x_{12}}(-2\tau_1\tau_1' + x_{22}\tau_1^2 + x_{11}\tau'^2_1)\right\}. \tag{23}$$

In the case of the simple harmonic oscillator, the parameters k_{jk} are

$$k_{11} = \frac{1}{2m}\,, \quad k_{22} = \frac{1}{2}m\omega^2\,, \quad k_{21} = 0\,. \tag{24}$$

We can easily find the functions f_{jk}, and the propagator results from Eq. (22).

$$G_1(\tau_1, \tau_1', t) = \sqrt{\frac{i\Omega}{2\,\sin(\Omega t)}}\exp\left\{-\frac{i\Omega}{2\,\sin(\Omega t)}\left(-2\tau_1\tau_1' + (\tau_1^2 + \tau'^2_2)\cos(\Omega t)\right)\right\}, \tag{25}$$

where $m = 1$, $\hbar = 1$.

We have calculated the above well-known propagator to illustrate the method.[10,11] We shall use the same method for more complicated systems in the next section.

3. Two Coupled Systems

In this section, we will study the harmonic oscillator in two dimensions in noncommuting space.[12] We will use the following Hamiltonian

$$\mathcal{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m\omega_1^2\hat{q}_1^2 + \frac{1}{2}m\omega_2^2\hat{q}_2^2\,.$$

We will transform the problem of the noncommuting two-dimensional space to a problem of two coupled harmonic oscillators in the usual quantum mechanical space.

For this purpose, we perform the following linear transformations

$$\begin{aligned} \hat{P}_1 &= \hat{p}_1 & \hat{Q}_1 &= \hat{q}_1 \\ \hat{P}_2 &= \hat{p}_2 + \frac{\lambda}{\hbar}\hat{q}_1\,, & \hat{Q}_2 &= \hat{q}_2 - \frac{\theta}{\hbar}\hat{p}_1\,. \end{aligned} \tag{26}$$

The commutators relations for the basic operators with the capital letters are

$$\begin{aligned} &[\hat{Q}_1, \hat{Q}_2] = 0\,, \quad [\hat{P}_1, \hat{P}_2] = 0\,, \\ &[\hat{Q}_1, \hat{P}_1] = i\hbar\,, \quad [\hat{Q}_2, \hat{P}_2] = i\hbar\left(1 - \frac{\lambda\theta}{\hbar^2}\right). \end{aligned} \tag{27}$$

The basic operators of the second dimension $[\hat{P}_2, \hat{Q}_2]$ satisfy a deformed commutation relation.[13] We denote the deformed parameter by μ, that is

$$\mu = 1 - \frac{\lambda\theta}{\hbar^2}\,. \tag{28}$$

The μ parameter has three critical values, $\mu = 0$ and $\mu = \pm 1$. It becomes unity if $\lambda = 0$ or $\theta = 0$.

For the case where

$$\frac{\lambda}{\hbar} = \frac{\hbar}{\theta}\,, \tag{29}$$

the μ parameter vanishes and the operators $\hat{P}_2$ and $\hat{Q}_2$ commute.

In the case where

$$\frac{\lambda}{\hbar} = 2\frac{\hbar}{\theta}\,, \tag{30}$$

the parameter $\mu = -1$ and the last commutation relation becomes the complex conjugate of the standard one. This case can be treated if we change the operators as for instance $\hat{P}_2 \to \hat{P}_2$ and $\hat{Q}_2 \to -\hat{Q}_2$.

With these operators, the Hamiltonian becomes

$$\begin{aligned} \mathcal{H}(\hat{\mathbf{P}}, \hat{\mathbf{Q}}) = {} & \frac{1}{2m}\left(1 + m^2\omega_2^2\frac{\theta^2}{\hbar^2}\right)\hat{P}_1^2 + \frac{1}{2}m\left(\omega_1^2 + \frac{\lambda^2}{m^2\hbar^2}\right)\hat{Q}_1^2 \\ & + \frac{1}{2m}\hat{P}_2^2 + \frac{1}{2}m\omega_2^2\hat{Q}_2^2 + m\omega_2^2\frac{\theta}{\hbar}\hat{P}_1\hat{Q}_2 - \frac{\lambda}{m\hbar}\hat{P}_2\hat{Q}_1\,. \end{aligned} \tag{31}$$

This is a Hamiltonian of two harmonic oscillators in the usual quantum mechanical space. The last two factors of the above Hamiltonian are the coupling terms.

To simplify the relations, we shall use the following symbolism

$$\hat{P}_1 \to \hat{T}_1\,, \quad \hat{Q}_1 \to \hat{T}_2\,, \quad \hat{P}_2 \to \hat{T}_3\,, \quad \hat{Q}_2 \to \hat{T}_4\,. \tag{32}$$

The commutation relations become

$$[\hat{T}_2, \hat{T}_1] = c_{21}\,, \quad [\hat{T}_4, \hat{T}_3] = c_{43} = c_{21}\mu\,, \quad c_{21} = i\hbar\,, \tag{33}$$

while all the other commutators vanish. The Hamiltonian is written as

$$\mathcal{H}(\hat{\mathbf{T}}) = k_{11}\hat{T}_1^2 + k_{22}\hat{T}_2^2 + k_{33}\hat{T}_3^2 + k_{44}\hat{T}_4^2 + k_{41}\hat{T}_4\hat{T}_1 + k_{32}\hat{T}_3\hat{T}_2 \,. \tag{34}$$

Where we have set

$$\begin{aligned} k_{11} &= \frac{1}{2m}\left(1 + m^2\omega_2^2\frac{\theta^2}{\hbar^2}\right), \quad k_{22} = \frac{1}{2}m\omega_1^2\left(1 + \frac{\lambda^2}{m^2\omega_1^2\hbar^2}\right), \\ k_{33} &= \frac{1}{2m}\,, \quad k_{44} = \frac{1}{2}m\omega_2^2\,, \quad k_{32} = -\frac{\lambda}{m\hbar}\,, \quad k_{41} = m\omega_2^2\frac{\theta}{\hbar}\,. \end{aligned} \tag{35}$$

The coupling terms k_{32} and k_{41} become zero in the case where $\lambda \to 0$ and $\theta \to 0$.

4. Canonical Order of the Time Evolution Operator

We will expand the time evolution operator in an appropriate ordered form so that the propagator will be calculated easily in a straight manner.

We look for the following expansion.

$$\begin{aligned} U(t) = {} & e^{f_{44}T_4^2}e^{\frac{1}{2}f_{43}(T_4T_3+T_3T_4)}e^{f_{42}T_4T_2}e^{f_{41}T_4T_1}e^{f_{33}T_3^2}e^{f_{32}T_3T_2}e^{f_{31}T_3T_1} \\ & \times e^{f_{22}T_2^2}e^{\frac{1}{2}f_{21}(T_2T_1+T_1T_2)}e^{f_{11}T_1^2}\,, \end{aligned}$$

where the functions $f_{jk}(t)$ are time dependent and obviously satisfy the initial conditions

$$f_{jk}(0) = 0\,.$$

The operator is written as a product of four operators.

$$U(t) = U_4(\hat{T}_4, \hat{T}_3, \hat{T}_2, \hat{T}_1, t)U_3(\hat{T}_3, \hat{T}_2, \hat{T}_1, t)U_2(\hat{T}_2, \hat{T}_1, t)U_1(\hat{T}_1, t)\,. \tag{36}$$

The operator U_2U_1 depends only on the basic operators of the first dimension, so the propagator of the system is

$$G(\tau_2, \tau_2', \tau_1, \tau_1', t) = U_4U_3\delta(\tau_2 - \tau_2')U_2U_1\delta(\tau_1 - \tau_1')\,. \tag{37}$$

In the sequel, we will find the unknown functions f_{jk}.

We differentiate with respect to time and find the operator

$$\left(i\hbar\frac{\partial}{\partial t}U(t)\right)U(t)^{-1}\,,$$

which is equal to the Hamiltonian of the system. In fact, we have

$$i\hbar\frac{\partial}{\partial t}U(t) = \mathcal{H}U(t) \;\Rightarrow\; \left(i\hbar\frac{\partial}{\partial t}U(t)\right)U(t)^{-1} = \mathcal{H}\,. \tag{38}$$

The calculations proceed as follows

$$\begin{aligned} \mathcal{H} &= \left(i\hbar\frac{\partial}{\partial t}(U_4U_3)\right)(U_4U_3)^{-1} + (U_4U_3)\left(i\hbar\frac{\partial}{\partial t}(U_2U_1)\right)(U_2U_1)^{-1}(U_4U_3)^{-1} \\ &= f_{44}'T_4^2 + \frac{1}{2}f_{43}'e^{f_{44}T_4^2}(T_4T_3 + T_3T_4)e^{-f_{44}T_4^2} \end{aligned}$$

$$
\begin{aligned}
&+ f'_{42} e^{f_{44}T_4^2} e^{(1/2)f_{43}(T_4T_3+T_3T_4)} T_4 T_2 e^{-(1/2)f_{43}(T_4T_3+T_3T_4)} e^{-f_{44}T_4^2} \\
&+ f'_{41} e^{f_{44}T_4^2} e^{(1/2)f_{43}(T_4T_3+T_3T_4)} e^{f_{42}T_4T_2} T_4 T_1 e^{-f_{42}T_4T_2} e^{-(1/2)f_{43}(T_4T_3+T_3T_4)} e^{-f_{44}T_4^2} \\
&+ U_4[f'_{33}T_3^2 + f'_{32} e^{f_{33}T_3^2} T_3 T_2 e^{-f_{33}T_3^2} \\
&+ f'_{31} e^{f_{33}T_3^2} e^{f_{32}T_3T_2} T_3 T_1 e^{-f_{32}T_3T_2} e^{-f_{33}T_3^2}] U_4^{-1} \\
&+ U_4 U_3 \left[f'_{22} T_2^2 + \frac{1}{2} f'_{21} e^{f_{22}T_2^2} (T_2T_1 + T_1T_2) e^{-f_{22}T_2^2} + f'_{11} U_2 T_1^2 U_2^{-1} \right] U_3^{-1} U_4^{-1} \,.
\end{aligned} \tag{39}
$$

We rearrange the above second-order polynomial with the help of standard operator identities, and write it as a linear combination of the operators $T_j T_k$. We equate this operator with the Hamiltonian and finally find a first-order differential system with ten equations and ten unknown functions, f_{jk}.

We write the system in two groups. The first group contains only the unknown functions which correspond to the first dimension space. This group is

$$
\begin{aligned}
i\hbar f'_{22} &= 4c_{21}^2 f_{22}^2 k_{11}(t) - 2c_{21} f_{22} k_{21}(t) + k_{22}(t) \,, \\
i\hbar f'_{21} &= -4c_{21} f_{22} k_{11}(t) + k_{21}(t) \,, \\
i\hbar f'_{11} &= e^{-2c_{21}f_{21}} k_{11}(t) \,,
\end{aligned} \tag{40}
$$

where we have set

$$
\begin{aligned}
k_{22}(t) &= k_{22} - c_{43} f_{42} e^{-c_{43}f_{43}} k_{32} + c_{43}^2 f_{42}^2 e^{-2c_{43}f_{43}} k_{33} \,, \\
k_{21}(t) &= -c_{43} f_{41} e^{-c_{43}f_{43}} (k_{32} - 2c_{43} f_{42} e^{-c_{43}f_{43}} k_{33}) \,, \\
k_{11}(t) &= k_{11} + c_{43}^2 f_{41}^2 e^{-2c_{43}f_{43}} k_{33} \,.
\end{aligned} \tag{41}
$$

Notice that these equations are similar to the corresponding system of the one-dimensional case [Eq. (15)], but now the constants k_{22}, k_{21}, and k_{11} are a function of the time.

The rest of the seven equations are

$$
\begin{aligned}
i\hbar f'_{33} &= e^{-2c_{43}f_{43}} k_{33} + 2c_{21} c_{43} f_{32} f_{41} e^{-2c_{43}f_{43}} k_{33} + c_{21}^2 f_{32}^2 k_{11}(t) \\
&\quad - c_{21}^2 f_{31}^2 k_{22}(t) \,, \\
i\hbar f'_{32} &= e^{-c_{43}f_{43}} k_{32} - 2c_{43} f_{42} e^{-2c_{43}f_{43}} k_{33} + c_{21} c_{43} f_{32} k_{21}(t) + 2c_{21} f_{31} k_{22}(t) \,, \\
i\hbar f'_{31} &= -2c_{43} f_{41} e^{-2c_{43}f_{43}} k_{33} - 2c_{21} f_{32} k_{11}(t) - c_{21} c_{43} f_{31} k_{21}(t) \,,
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
i\hbar f'_{44} &= 4c_{43}^2 f_{44}^2 k_{33} + k_{44} - c_{21} f_{42} e^{-c_{43}f_{43}} k_{41} \\
&\quad + c_{21}^2 f_{42}^2 e^{-2c_{43}f_{43}} k_{11} - c_{21}^2 f_{41}^2 e^{-2c_{43}f_{43}} k_{22}(t) \,, \\
i\hbar f'_{43} &= -4c_{43} f_{44} k_{33} + c_{21} f_{41} e^{-c_{43}f_{43}} k_{21}(t) \,, \\
i\hbar f'_{42} &= 2k_{22} c_{21} f_{41} - 2c_{43} e^{c_{43}f_{43}} f_{44} k_{32} - c_{43} c_{21} f_{41} f_{42} e^{-c_{43}f_{43}} k_{32} \,, \\
i\hbar f'_{41} &= -2c_{21} f_{42} k_{11} + e^{c_{43}f_{43}} k_{41} \,.
\end{aligned} \tag{43}
$$

The last four differential equations contain only the four unknown functions f_{44}, f_{43}, f_{42}, and f_{41}, and can be solved easily. As for the one-dimensional case, we set

$$f_{44} = \frac{1}{2c_{43}}\left(\frac{x_{43}}{x_{33}} - \frac{1}{\mu}\frac{x_{13}}{x_{33}}\frac{x_{23}}{x_{33}}\right), \quad f_{42} = \frac{1}{c_{43}}x_{23}\,,$$

$$f_{41} = \frac{1}{c_{43}}x_{13}\,, \quad f_{43} = \frac{1}{c_{43}}\log x_{33}\,.$$

We can prove that the new unknown functions x_{13}, x_{23}, x_{33}, and x_{43} satisfy the following differential system

$$\begin{aligned} x'_{1j} &= -2k_{11}x_{2j} + \mu\, k_{41}x_{3j} \\ x'_{2j} &= 2k_{22}x_{1j} - \mu\, k_{32}x_{4j}\,, \\ x'_{3j} &= k_{32}x_{1j} - 2\mu\, k_{33}x_{4j}\,, \\ x'_{4j} &= -k_{41}x_{2j} + 2\mu\, k_{44}x_{3j}\,, \end{aligned} \tag{44}$$

where $j = 3$. The unknown functions $x_{n3}(t)$ satisfy the initial conditions

$$x_{13}(0) = x_{23}(0) = x_{43}(0) = 0\,, \quad x_{33}(0) = 1\,. \tag{45}$$

The functions x_{nj} give the time evolution of the basic operators $\hat{T}_j$. We have

$$\hat{T}_n(t) = e^{-\frac{i}{\hbar}t\mathcal{H}}\hat{T}_j(0)e^{\frac{i}{\hbar}t\mathcal{H}} = x_{nj}(t)\hat{T}_j(0)\,. \tag{46}$$

The functions x_{nj} satisfy the same Eq. (44) where $j = 1,2,3,4$ with the initial conditions

$$x_{nj}(0) = \delta_{nj}\,. \tag{47}$$

Because the commutation relations of the operators $\hat{T}_j$ are conservative, that is

$$[\hat{T}_j(t), \hat{T}_k(t)] = [\hat{T}_j, \hat{T}_k]\,, \tag{48}$$

the functions x_{nj} satisfy the equations

$$\begin{aligned} -x_{14}x_{23} + x_{13}x_{24} + \mu(-x_{34}x_{43} + x_{33}x_{44}) &= \mu\,, \\ -x_{14}x_{22} + x_{12}x_{24} + \mu(-x_{34}x_{42} + x_{32}x_{44}) &= 0\,, \\ -x_{13}x_{22} + x_{12}x_{23} + \mu(-x_{33}x_{42} + x_{32}x_{43}) &= 0\,, \\ -x_{14}x_{21} + x_{11}x_{24} + \mu(-x_{34}x_{41} + x_{31}x_{44}) &= 0\,, \\ -x_{13}x_{21} + x_{11}x_{23} + \mu(-x_{33}x_{41} + x_{31}x_{43}) &= 0\,, \\ -x_{12}x_{21} + x_{11}x_{22} + \mu(-x_{32}x_{41} + x_{31}x_{42}) &= 1\,. \end{aligned} \tag{49}$$

We can prove that all the unknown functions f_{jk} can be written with the help of the functions $x_{jk}(t)$.

We set

$$a_1 = \frac{x_{13}}{x_{33}}\,, \quad a_2 = \frac{x_{23}}{x_{33}}\,, \tag{50}$$

and we find

$$f_{11} = -\frac{1}{2c_{21}} \frac{x_{12} - a_1 x_{32}}{x_{11} - a_1 x_{31}}, \quad f_{22} = \frac{1}{2c_{21}} \frac{x_{21} - a_2 x_{31}}{x_{11} - a_1 x_{31}},$$
$$f_{21} = \frac{1}{c_{21}} \log(x_{11} - a_1 x_{31}), \tag{51}$$

$$f_{33} = -\frac{1}{2c_{43}} \left(\frac{x_{34}}{x_{33}} + \frac{1}{\mu}(x_{14} - a_1 x_{34})(x_{24} - a_2 x_{34}) \right),$$
$$f_{31} = -\frac{1}{c_{43}}(x_{14} - a_1 x_{34}), \quad f_{32} = -\frac{1}{c_{43}}(x_{24} - a_2 x_{34}), \tag{52}$$

$$f_{44} = \frac{1}{2c_{43}} \left(\frac{x_{43}}{x_{33}} - \frac{1}{\mu} \frac{x_{13}}{x_{33}} \frac{x_{23}}{x_{33}} \right), \quad f_{42} = \frac{1}{c_{43}} x_{23},$$
$$f_{41} = \frac{1}{c_{43}} x_{13}, \quad f_{43} = \frac{1}{c_{43}} \log x_{33}. \tag{53}$$

In the case where the coordinates and momenta operators commute, then

$$\lambda = 0, \quad \theta = 0 \Rightarrow k_{41} = 0, \quad k_{32} = 0, \quad c_{21} = c_{43}. \tag{54}$$

The functions a_1 and a_2 vanish and the solution becomes

$$f_{44} = \frac{1}{2c_{21}} \frac{x_{43}}{x_{33}}, \quad f_{43} = \frac{1}{c_{21}} \log x_{33}, \quad f_{33} = -\frac{1}{2c_{21}} \frac{x_{34}}{x_{33}}, \tag{55}$$

$$f_{22} = \frac{1}{2c_{21}} \frac{x_{21}}{x_{11}}, \quad f_{21} = \frac{1}{c_{21}} \log x_{11}, \quad f_{11} = -\frac{1}{2c_{21}} \frac{x_{12}}{x_{11}}. \tag{56}$$

Obviously, the problem is now reduced to that of two independent simple harmonic oscillators.

5. The Exact Propagator of the System

In this section, we will calculate the exact propagator of the system. As is well-known, the action of the time evolution operator on the delta function gives the propagator of the system.

$$G(\tau_1, \tau_1', \tau_2, \tau_2', t) = U(t)\delta(\tau_1 - \tau_1')\delta(\tau_2 - \tau_2'). \tag{57}$$

We will first examine the case where $c_{43} \neq 0$.

Because of the commutation relations (33), only two quantities can be simultaneously measured. We choose the following observables

$$T_2 = Q_1 = q_1 \to \tau_1, \quad T_4 = Q_2 = q_2 - \frac{\theta}{\hbar} p_1 \to \tau_2. \tag{58}$$

For the calculations, we consider the following representation

$$T_1 = -c_{21}\partial_{\tau_1}, \quad T_2 = \tau_1, \quad T_3 = -c_{43}\partial_{\tau_2}, \quad T_4 = \tau_2. \tag{59}$$

We can of course choose another couple of two commuting observables. The various propagators are appropriate Fourier transforms of each other.

The propagator for the above representation is

$$\begin{aligned} &G(\tau_1, \tau_1', \tau_2, \tau_2', t) \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_2 dk_1 U_4 U_3 U_2 U_1 e^{ik_1(\tau_1 - \tau_1')} e^{ik_2(\tau_2 - \tau_2')} \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_2 dk_1 e^{is_1 k_1^2 + is_2 k_2^2 + is_3 k_1 k_2 + ik_1(\tau_1(t) - \tau_1') + ik_2(\tau_2(t) - \tau_2') + u_0} . \end{aligned} \tag{60}$$

The functions s_1, s_2 play the main role for the integration with respect to k_1 and k_2. If one of these parameters is zero, the integration leaves a corresponding delta function to the result. These are functions of f_{ij} and have the form:

$$\begin{aligned} &s_1 = ic_{21}^2 f_{11} , \quad s_2 = ic_{43}^2 (f_{33} + c_{21}^2 f_{22} f_{31}^2) , \quad s_3 = ic_{21} c_{43} e^{-c_{21} f_{21}} f_{31} , \\ &\tau_1(t) = e^{-c_{21} f_{21}} (\tau_1 - c_{21} e^{-c_{43} f_{43}} f_{41} \tau_2) , \\ &\tau_2(t) = e^{-c_{43} f_{43}} \tau_2 - c_{43} (f_{32} - 2c_{21} f_{22} f_{31})(\tau_1 - c_{21} f_{41} e^{-c_{43} f_{43}} \tau_2) , \\ &u_0 = f_{44} \tau_2^2 + f_{42} e^{-c_{43} f_{43}} \tau_1 \tau_2 + f_{22} (\tau_1 - c_{21} f_{41} e^{-c_{43} f_{43}} \tau_2)^2 . \end{aligned} \tag{61}$$

Under certain conditions, the integration of the formula (60) gives the propagator of the system. We find

$$\begin{aligned} G(\tau_1, \tau_1', \tau_2, \tau_2', t) = {} & \frac{1}{\sqrt{e^{c_{21} f_{21} + c_{43} f_{43}} (s_3^2 - 4 s_1 s_2)}} \\ & \times \exp\left\{ \frac{i}{s_3^2 - 4 s_1 s_2} (s_2 (\tau_1(t) - \tau_1')^2 - s_3 (\tau_1(t) - \tau_1')(\tau_2(t) - \tau_2') \right. \\ & \left. + s_1 (\tau_2(t) - \tau_2')^2) + u_0 \right\} . \end{aligned} \tag{62}$$

With respect to the functions x_{jk}, the propagator becomes

$$\begin{aligned} G(\tau_1, \tau_1', \tau_2, \tau_2', t) = {} & \frac{1}{\sqrt{s_0}} \exp\left\{ -\frac{1}{2 s_0} [c_{43} (2 \tau_1 \tau_1' x_{34} + {\tau'}_1^2 (x_{14} x_{31} - x_{11} x_{34}) \right. \\ & + \tau_1^2 (x_{24} x_{32} - x_{22} x_{34})) + c_{21} (2 \tau_2 \tau_2' x_{12} + {\tau'}_2^2 (x_{13} x_{32} - x_{12} x_{33}) \\ & + \tau_2^2 (x_{14} x_{42} - x_{12} x_{44})) - 2 c_{21} \tau_1' (\tau_2 x_{14} - \tau_2' (x_{14} x_{33} - x_{13} x_{34})) \\ & \left. - 2 c_{43} \tau_1 (\tau_2' x_{32} + \tau_2 (x_{34} x_{42} - x_{32} x_{44}))] \right\} , \end{aligned} \tag{63}$$

where

$$s_0 = c_{21} c_{43} (-x_{14} x_{32} + x_{12} x_{34}) . \tag{64}$$

If we find the propagator, we can calculate the time evolution of a quantum system. The initial state must have the following form.

$$\psi(\tau_1, \tau_2, 0) = \psi_0\left(q_1, q_2 - \frac{\theta}{\hbar}p_1\right). \tag{65}$$

We can have an initial state as well, which may be a function of the commutative observables p_1 and q_2. That is

$$\psi(p_1, q_2, 0) = \psi_0\left(p_1, \tau_2 + \frac{\theta}{\hbar}p_1\right). \tag{66}$$

The propagator for this initial state is the Fourier transform of $G(\tau_1, \tau_1', \tau_2, \tau_2', t)$ with respect to the first dimension.

In the sequel, we will examine the case where

$$c_{43} = 0 \Rightarrow \mu = 0 \Rightarrow \frac{\lambda}{\hbar} = \frac{\hbar}{\theta}. \tag{67}$$

In the space of the T_j operators, we now have three commutative observables, and we choose the following

$$q_1 \to \tau_1, \quad p_2 + \frac{\lambda}{\hbar}q_1 \to \pi_2, \quad q_2 - \frac{\theta}{\hbar}p_1 \to \tau_2. \tag{68}$$

To calculate the propagator, we assume the representation

$$T_1 = -c_{21}\partial_{\tau_1}, \quad T_2 = \tau_1, \quad T_3 = \pi_2, \quad T_4 = \tau_2. \tag{69}$$

Another way to find the propagator is to first calculate the propagator in the first dimension, that is

$$G_1(\tau_1, \tau_1', t) = U_2U_1\delta(\tau_1 - \tau_1'). \tag{70}$$

The propagator can be found by the action of the operator

$$U_4U_3\delta(\tau_2 - \tau_2'), \tag{71}$$

on the distribution $G_1(\tau_1, \tau_1', t)$.

For the case where $c_{43} \neq 0$, this operator contains first- and second-order derivatives with respect to τ_2. We only find the action of these second-order derivatives because these operators drastically change the distribution $G_1(\tau_1, \tau_1', t)$. We find

$$\begin{aligned} G(\tau_1, \tau_1', \tau_2, \tau_2', t) &= e^{f_{44}\tau_2^2}e^{-(1/2)c_{43}f_{43}(\tau_2\partial\tau_2 + (\partial\tau_2)\tau_2)}e^{f_{42}\tau_2\tau_1}e^{-c_{21}f_{41}\tau_2\partial_{\tau_1}}e^{-c_{43}f_{32}\tau_1\partial_{\tau_2}} \\ &\quad \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-c_{43}^2f_{33}k^2 + ik(\tau_2 - \tau_2')}\, G_1(\tau_1 + ikc_{43}c_{21}f_{31}, \tau_1', t)\, dk. \end{aligned} \tag{72}$$

For the present case where $c_{43} = 0$, the operator $U_4U_3\delta(\tau_2 - \tau_2')$ does not contain any operator with respect to the second dimension. After a simple calculation, we find

$$\begin{aligned}G(\tau_1,\tau_1',\tau_2,\tau_2',t) &= U_4 U_3 \delta(\tau_2-\tau_2') G_1(\tau_1,\tau_1',t)\\ &= e^{f_{44}\tau_2^2+f_{33}\pi_2^2+(f_{43}-c_{21}f_{32}f_{41})\pi_2\tau_2} e^{\tau_1(f_{32}\pi_2+f_{42}\tau_2)}\\ &\quad\times G_1\left(\tau_1-c_{21}(f_{31}\pi_2+f_{41}\tau_2),\tau_1',t\right)\delta(\tau_2-\tau_2')\,.\end{aligned} \tag{73}$$

6. The Propagator of the Harmonic Oscillator

In this section, we will solve the system (44) when the k_{jk} parameters of the Hamiltonian are time-independent.

We write the system in matrix form

$$\frac{d}{d\,t}X(t) = \begin{pmatrix} 0 & -2k_{11} & \mu k_{41} & 0 \\ 2k_{22} & 0 & 0 & -\mu k_{32} \\ k_{32} & 0 & 0 & -2\mu k_{33} \\ 0 & -k_{41} & 2\mu k_{44} & 0 \end{pmatrix} X(t)\,, \tag{74}$$

with initial conditions $x_{jk}(0)=\delta_{jk}$.

The solution is the following

$$\begin{aligned} x_{23} &= -\mu x_{41} = \mu r_1 g_3\,, & x_{14} &= -\mu x_{32} = \mu r_2 g_3\,,\\ x_{11} &= x_{22} = g_4 - r_3 g_3\,, & x_{33} &= x_{44} = g_4 + r_3 g_3\,,\\ x_{31} &= k_{32}(g_2-r_3g_1)+2\mu k_{33}r_1g_1\,, & x_{13} &= \mu k_{41}(g_2+r_3g_1)-2\mu k_{11}r_1g_1\,,\\ x_{42} &= -k_{41}(g_2-r_3g_1)-2\mu k_{44}r_2g_1\,, & x_{24} &= -\mu k_{32}(g_2+r_3g_1)+2\mu k_{22}r_2g_1\,,\\ x_{21} &= 2k_{22}(g_2-r_3g_1)+\mu k_{32}r_1g_1\,, & x_{34} &= -2\mu k_{33}(g_2+r_3g_1)+\mu k_{32}r_2g_1\,,\\ x_{12} &= -2k_{11}(g_2-r_3g_1)-\mu k_{41}r_2g_1\,, & x_{43} &= 2\mu k_{44}(g_2+r_3g_1)-\mu k_{41}r_1g_1\,, \end{aligned} \tag{75}$$

where the constants r_1,r_2, and r_3 are

$$\begin{gathered} r_1 = 2(k_{22}k_{41}-k_{32}k_{44}\mu)\,,\quad r_2 = 2(k_{11}k_{32}-k_{33}k_{41}\mu)\,,\\ r_3 = 2(k_{11}k_{22}-k_{33}k_{44}\mu^2)\,, \end{gathered} \tag{76}$$

and the functions $g_j(t)$, $j=1,2,3,4$ are

$$g_1(t) = \frac{1}{\Omega_1^2-\Omega_2^2}\left(\frac{\sin\,\Omega_1 t}{\Omega_1}-\frac{\sin\,\Omega_2 t}{\Omega_2}\right),\quad g_2(t) = \frac{1}{2}\left(\frac{\sin\,\Omega_1 t}{\Omega_1}+\frac{\sin\,\Omega_2 t}{\Omega_2}\right), \tag{77}$$

$$g_3(t) = \frac{1}{\Omega_1^2-\Omega_2^2}\left(\cos\,\Omega_1 t-\cos\,\Omega_2 t\right),\quad g_4(t) = \frac{1}{2}\left(\cos\,\Omega_1 t+\cos\,\Omega_2 t\right). \tag{78}$$

The angles Ω_1 and Ω_2 are

$$\begin{aligned} \Omega_1 &= \sqrt{w_2+\sqrt{w_2^2-w_1^2}} = \sqrt{\frac{w_2+w_1}{2}}+\sqrt{\frac{w_2-w_1}{2}}\,,\\ \Omega_2 &= \sqrt{w_2-\sqrt{w_2^2-w_1^2}} = \sqrt{\frac{w_2+w_1}{2}}-\sqrt{\frac{w_2-w_1}{2}}\,, \end{aligned} \tag{79}$$

where

$$w_1 = \mu\sqrt{k_{32}^2 - 4k_{22}k_{33}}\sqrt{k_{41}^2 - 4k_{11}k_{44}}\,,$$
$$w_2 = -2k_{11}k_{22} + \mu k_{32}k_{41} - 2\mu^2 k_{33}k_{44}\,. \tag{80}$$

We substitute the constants k_{ij} and find

$$w_1 = \omega_1\omega_2\left(-1 + \frac{\lambda\theta}{\hbar^2}\right), \quad w_2 = \frac{\omega_1^2}{2} + \frac{\omega_2^2}{2} + \frac{1}{2}m^2\omega_1^2\omega_2^2\frac{\theta^2}{\hbar^2} + \frac{\lambda^2}{2m^2\hbar^2}\,. \tag{81}$$

An interesting limiting case which cannot be found from the final result is the case where one of the angles becomes zero. This happens if

$$w_1 = 0\,. \tag{82}$$

This parameter is zero if one of the frequencies ω_1 or ω_2 is zero. This case is essentially a one-dimensional problem and it will be studied in another paper.[14] The value of w_1 is also zero when the parameter μ vanishes.

$$\mu = 0 \;\Rightarrow\; \frac{\lambda}{\hbar} = \frac{\hbar}{\theta}\,. \tag{83}$$

In this case, the angle $\Omega_2 = 0$ while the second angle is

$$\Omega_1 = \sqrt{4k_{11}k_{22}} = \sqrt{\omega_1^2 + \omega_2^2 + m^2\omega_1^2\omega_2^2\frac{\theta^2}{\hbar^2} + \frac{\lambda^2}{m^2\hbar^2}}\,. \tag{84}$$

Now, because of the relation $c_{43} = c_{21}\mu$, the commutation relation between the operators T_4 and T_3 vanish. We have a decoherence situation in the second dimension. This very interesting case will be studied separately. This second angle may also be zero only for complex frequencies or mass. In this case, there are no periodic terms at all in the final result.

In the sequel, we will calculate the propagator for the following variables

$$m = 1\,, \quad \hbar = 1\,. \tag{85}$$

The angles Ω_1 and Ω_2 are

$$\Omega_1 = \frac{1}{2}\left(\sqrt{(\omega_1+\omega_2)^2 + (-\omega_1\,\omega_2\theta + \lambda)^2} + \sqrt{(\omega_1-\omega_2)^2 + (\omega_1\omega_2\theta + \lambda)^2}\right),$$
$$\Omega_2 = \frac{1}{2}\left(\sqrt{(\omega_1+\omega_2)^2 + (-\omega_1\omega_2\theta + \lambda)^2} - \sqrt{(\omega_1-\omega_2)^2 + (\omega_1\omega_2\theta + \lambda)^2}\right). \tag{86}$$

In the limit where $\lambda \to 0$ and $\theta \to 0$, we have $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Our calculations give the propagator

$$G(\tau_1, \tau_1', \tau_2, \tau_2', t) = \frac{\Omega_1^2 - \Omega_2^2}{2\sqrt{-z_1 z_2}}\sqrt{\frac{\omega_2}{\omega_1\mu}}$$
$$\times\ \exp\left\{i\frac{\Omega_1^2 - \Omega_2^2}{4z_1 z_2}\frac{\omega_2}{\omega_1}(\tau_2\tau_1' - \tau_1\tau_2')(\lambda + \omega_2^2\theta)(\cos(\Omega_1 t) - \cos(\Omega_2 t))\right.$$

$$
\begin{aligned}
&-\frac{i}{4z_1z_2}\frac{\omega_2}{\omega_1}(\tau_1\tau_2-\tau_2'\tau_1')\Bigg[\left(\Omega_1^2+\Omega_2^2-2\frac{\omega_2}{\omega_1}\Omega_1\Omega_2\right)\\
&\times(\lambda+\omega_2^2\theta)(-1+\cos(\Omega_1t)\cos(\Omega_2t))\\
&-\frac{\omega_2}{\omega_1}\Bigg(\left(\Omega_1^2+\Omega_2^2-2\frac{\omega_1}{\omega_2}\Omega_1\Omega_2\right)(\lambda+\omega_2^2\theta)\\
&-(\Omega_1^2-\Omega_2^2)^2\theta\Bigg)\sin(\Omega_1t)\sin(\Omega_2t)\Bigg]\Bigg\}\\
&\times\ \exp\Bigg\{i\frac{\Omega_1^2-\Omega_2^2}{8z_1z_2}\Bigg[\Omega_1\left(1+\omega_2^2\theta^2-\frac{\Omega_2^2}{\omega_1^2}\right)\\
&\times\Bigg(\frac{\omega_2}{\omega_1\mu}\Big(-2\tau_2\tau_2'+(\tau_2^2+{\tau'}_2^2)\cos(\Omega_2t)\Big)\sin(\Omega_1t)\\
&+\Big(-2\tau_1\tau_1'+(\tau_1^2+{\tau'}_1^2)\cos(\Omega_1t)\Big)\sin(\Omega_2t)\Bigg)\\
&-\Omega_2\left(1+\omega_2^2\theta^2-\frac{\Omega_1^2}{\omega_1^2}\right)\Big((-2\tau_1\tau_1'+(\tau_1^2+{\tau'}_1^2)\cos(\Omega_2t))\sin(\Omega_1t)\\
&+\frac{\omega_2}{\omega_1\mu}\Big(-2\tau_2\tau_2'+(\tau_2^2+{\tau'}_2^2)\cos(\Omega_1t)\Big)\sin(\Omega_2t)\Big)\Bigg]\Bigg\},
\end{aligned}\tag{87}
$$

where

$$
\begin{aligned}
z_1&=\Omega_2\left(1+\omega_2^2\theta^2-\frac{\Omega_1^2}{\omega_1^2}\right)\cos\left(\frac{\Omega_1}{2}t\right)\sin\left(\frac{\Omega_2}{2}t\right)\\
&\quad-\Omega_1\left(1+\omega_2^2\theta^2-\frac{\Omega_2^2}{\omega_1^2}\right)\cos\left(\frac{\Omega_2}{2}t\right)\sin\left(\frac{\Omega_1}{2}t\right),
\end{aligned}\tag{88}
$$

$$
\begin{aligned}
z_2&=\Omega_2\left(1+\omega_2^2\theta^2-\frac{\Omega_1^2}{\omega_1^2}\right)\cos\left(\frac{\Omega_2}{2}t\right)\sin\left(\frac{\Omega_1}{2}t\right)\\
&\quad-\Omega_1\left(1+\omega_2^2\theta^2-\frac{\Omega_2^2}{\omega_1^2}\right)\cos\left(\frac{\Omega_1}{2}t\right)\sin\left(\frac{\Omega_2}{2}t\right).
\end{aligned}\tag{89}
$$

For the case of a free system,[15] then

$$
\omega_1=0\,,\quad \omega_2=0\,,\quad \Omega_1=\lambda\,,\quad \Omega_2=0\,,\tag{90}
$$

the propagator becomes

$$G(\tau_1,\tau_1',\tau_2,\tau_2',t) = \frac{i\lambda}{2\mu\sin\left(\frac{\lambda t}{2}\right)}\exp\left\{\frac{i\lambda}{2\mu}(\tau_1+\tau_1')(\tau_2-\tau_2') + \frac{1}{4}i\lambda\cot\left(\frac{\lambda t}{2}\right)\left((\tau_1-\tau_1')^2+\frac{1}{\mu^2}(\tau_2-\tau_2')^2\right)\right\}. \quad (91)$$

We can write the above formula equivalently as follows

$$G(\tau_1,\tau_1',\tau_2,\tau_2',t) = \frac{i\lambda}{2\mu\sin\left(\frac{\lambda t}{2}\right)}\exp\left\{\frac{i\lambda}{2\sin(\lambda t)}\left((\tau_1^2+{\tau'}_1^2)\cos(\lambda t)-2\tau_1\tau_1'\right) + \frac{i\lambda}{4\mu^2}\cot\left(\frac{\lambda t}{2}\right)\left(\tau_2-\tau_2'+\mu(\tau_1+\tau_1')\tan\left(\frac{\lambda t}{2}\right)\right)^2\right\}. \quad (92)$$

In the limit $\mu \to 0$, the above formula gives a delta function $\delta(\tau_2-\tau_2')$ in the second dimension. But the whole result is not correct, since in this limit, $c_{43} \to 0$ and the basic operators in the second dimension commute (see Eq. (111)).

As is well-known,[16] we can find the statistical distribution function from the propagator. The relation is

$$\rho(\tau_1,\tau_1',\tau_2,\tau_2',b) = G(\tau_1,\tau_1',\tau_2,\tau_2',-i\hbar t)\,. \quad (93)$$

The special case where $\tau_1=\tau_1'$ and $\tau_2=\tau_2'$, Eq. (87) gives

$$\rho(\tau_1,\tau_2,b) = \frac{\Omega_1^2-\Omega_2^2}{2\sqrt{z_1z_2}}\sqrt{\frac{\omega_2}{\omega_1\mu}} \times \exp\left\{-2(\Omega_1^2-\Omega_2^2)\sinh(f_1)\sinh(f_2)\left(\frac{1}{z_2}\tau_1^2+\frac{\omega_2}{\omega_1\mu}\frac{1}{z_1}\tau_2^2\right)\right\}, \quad (94)$$

where

$$f_1 = \frac{\hbar\Omega_1}{2}b = \frac{\hbar\Omega_1}{2kT}\,, \quad f_2 = \frac{\hbar\Omega_2}{2}b = \frac{\hbar\Omega_2}{2kT}\,. \quad (95)$$

This is the probability of finding the system at $(\tau_1,\,\tau_2) = (q_1,\,q_2-(\theta/\hbar)p_1)$. It is a product of two distinct Gaussian forms.

The partition function is

$$Z(b) = \iint \rho(\tau_1,\tau_2,b)d\tau_1 d\tau_2 = \frac{1}{4\,\sinh(f_1)\sin(f_2)}\,. \quad (96)$$

This partition function coincides with that of two independent harmonic oscillators with frequencies Ω_1 and Ω_2. So the energy eigenvalues of the system are

$$E_n = \hbar\Omega_1\left(n_1+\frac{1}{2}\right)+\hbar\Omega_2\left(n_2+\frac{1}{2}\right), \quad n_j = 0,1,2,3\ldots \quad (97)$$

We have started with a Hamiltonian with frequencies ω_1 and ω_2, and we finally find a partition function equivalent to that of a Hamiltonian with frequencies Ω_1 and Ω_2. The difference comes from the noncommutativity of coordinates and momentum. To justify this point, some authors[17] suggest the use of nonextensive statistics of Tsallis.[18]

The parameters λ and θ are functions of these four frequencies.

$$\begin{aligned}\lambda &= \frac{1}{2}\left(\sqrt{(\Omega_1-\Omega_2)^2-(\omega_1-\omega_2)^2}+\sqrt{(\Omega_1+\Omega_2)^2-(\omega_1+\omega_2)^2}\right),\\ \theta &= \frac{1}{2\omega_1\omega_2}\left(\sqrt{(\Omega_1-\Omega_2)^2-(\omega_1-\omega_2)^2}-\sqrt{(\Omega_1+\Omega_2)^2-(\omega_1+\omega_2)^2}\right).\end{aligned} \tag{98}$$

This system is equivalent to a system of two coupled harmonic oscillators in a constant magnetic field, in two dimensions[19] with commutation relations

$$[\hat{Q}_1,\hat{P}_1]=i\hbar\,,\quad [\hat{Q}_2,\hat{P}_2]=i\hbar\mu=i\hbar\frac{\Omega_1\Omega_2}{\omega_1\omega_2}\,. \tag{99}$$

The statistical distribution function, for the propagator (91) of free system, is

$$\rho(\tau_1,\tau_2,b)=\frac{\lambda}{2\mu\,\sinh\left(\frac{\lambda}{2kT}\right)}\,. \tag{100}$$

With an appropriate integration on a surface $\iint d\tau_1 d\tau_2 = \mu S$, we can find the partition function

$$z(b)=S\frac{\lambda}{2\,\sinh\left(\frac{\lambda}{2kT}\right)}\,. \tag{101}$$

This means that its energy eigenvalues are the usual Landau level eigenvalues.

7. Semi-Decoherent States

In this section, we will examine the case where

$$c_{43}=0\ \Rightarrow\ \mu=0\ \Rightarrow\ \frac{\lambda}{\hbar}=\frac{\hbar}{\theta}\,. \tag{102}$$

We can apply this problem to the case where we have a homogenous magnetic field parallel to the z-axes of the form

$$\mathbf{B}=(0,0,H)\,. \tag{103}$$

For this case, we set

$$\lambda=\hbar^2\frac{eH}{\hbar c}\,,\quad \theta=\frac{\hbar c}{eH}\,. \tag{104}$$

The initial commutation relations become

$$[q_1,p_1]=i\hbar\,,\quad [q_2,p_2]=i\hbar\,,\quad [p_1,p_2]=i\hbar^2\frac{eH}{\hbar c}\,,\quad [q_1,q_2]=i\frac{\hbar c}{eH}\,. \tag{105}$$

In this case, one of the angles is zero, and we assume that $\Omega_2=0$, $m=1$, and $\hbar=1$. So we find

$$\Omega_1=\frac{1}{\lambda}\sqrt{(\lambda^2+\omega_2^2)(\lambda^2+\omega_1^2)}\,. \tag{106}$$

In the space of the four operators T_j, we only have one commutation relation, while all the others vanish. So we have three commutative observables now, and we choose the following

$$q_1 \rightarrow \tau_1 \quad p_2 + \frac{\lambda}{\hbar} q_1 \rightarrow \pi_2 \,, \quad q_2 - \frac{\theta}{\hbar} p_1 \rightarrow \tau_2 \,. \tag{107}$$

To calculate the propagator, we assume the representation

$$T_1 = -c_{21}\partial_{\tau_1} \,, \quad T_2 = \tau_1 \,, \quad T_3 = \pi_2 \quad T_4 = \tau_2 \,. \tag{108}$$

The solution of the differential system (40)–(43) gives the unknown functions f_{jk}. With the help of the relation (73) and after some algebra, we finally find the propagator

$$\begin{aligned} G(\tau_1, \tau_1', \tau_2, \tau_2', \pi_2, t) = & \sqrt{\frac{-i\lambda^2\Omega_1}{2(\omega_2^2+\lambda^2)\sin(\Omega_1 t)}} \exp\left\{ -\frac{it}{2}\left(\frac{\omega_1^2}{\omega_1^2+\lambda^2}\pi_2^2 + \frac{\omega_2^2\lambda^2}{\omega_2^2+\lambda^2}\tau_2^2\right)\right. \\ & - \frac{i\omega_2^2\lambda}{\omega_2^2+\lambda^2}\tau_2(\tau_1-\tau_1') + \frac{i\lambda}{\Omega_1}\tan\left(\frac{\Omega_1}{2}t\right)\pi_2 \\ & \times \left(\tau_1+\tau_1' - \frac{\lambda}{\omega_1^2+\lambda^2}\pi_2\right) + \frac{i\lambda^2\Omega_1}{2(\omega_2^2+\lambda^2)\sin(\Omega_1 t)} \\ & \left. \times \left((\tau_1^2 + {\tau'}_1^2)\cos(\Omega_1 t) - 2\tau_1\tau_1'\right)\right\}\delta(\tau_2-\tau_2') \,. \end{aligned} \tag{109}$$

In the case of a free system, that is $\omega_1 = \omega_2 = 0$, then

$$\Omega_1 = \lambda \,, \tag{110}$$

and from the above propagator, we find

$$\begin{aligned} G(\tau_1, \tau_1', \tau_2, \tau_2', \pi_2, t) = & \sqrt{\frac{-i\lambda}{2\sin(\lambda t)}} \exp\left\{ \frac{i\lambda}{2\sin(\lambda t)}\left((\tau_1^2 + {\tau'}_1^2)\cos(\lambda t) - 2\tau_1\tau_1'\right)\right. \\ & \left. - \frac{i}{\lambda}\tan\left(\frac{\lambda t}{2}\right)\pi_2(\pi_2 - \lambda(\tau_1+\tau_1'))\right\}\delta(\tau_2-\tau_2') \,. \end{aligned} \tag{111}$$

This is the correct limit $\mu \to 0$ of the propagator (91).

8. Conclusion

In the present paper, we have examined the Harmonic Oscillator $\mathcal{H}$ in a constant magnetic field in two-dimensional noncommuting space. We have found the exact propagator of the system [Eq. (87)] which is the main goal of this paper. We first expand the time evolution operator in some kind of normal ordering. With the help of this formula, the propagator can be calculated easily and in a straightforward manner. We finally find the Boltzmann statistical density matrix and the partition function. The partition function is equivalent to that of two independent harmonic oscillators with two deformed frequencies Ω_1 and Ω_2.

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