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DEFORMED HARMONIC OSCILLATOR FOR NON-HERMITIAN OPERATOR AND THE BEHAVIOR OF PT AND CPT SYMMETRIES

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In the present paper we study the deformed harmonic oscillator for the non-Hermitian operator

$$\mathcal{H} = \frac{\alpha}{2m}\left(\hat{p}_1 + \frac{\lambda}{\hbar}\hat{q}_2\right)^2 + \frac{\beta m\omega^2}{2}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right)^2$$

where λ, θ are real positive parameters, since the parameters α, β, m are for the general case complex.

For the case $\alpha = 1$, $\beta = 1$ and mass m real, we find the eigenfunctions and eigenvalues of energy, the coherent states, the time evolution of the operators $\hat{q}_i$, $\hat{p}_j$ in the Heisenberg picture and the uncertainty relations. In this case the operator $\mathcal{H}$ is Hermitian and PT-symmetric. Also for the case m complex $\alpha = 1$, $\beta = 1$, the operator $\mathcal{H}$ is non-Hermitian and no more PT symmetric, but CPT symmetric with real discrete positive spectrum and the CPT symmetry is preserved. In the general case α, β, m complex, for the non-Hermitian operator $\mathcal{H}$, we obtain complex spectrum and for the special values of the complex parameters α, β the spectrum is real discrete and positive and the CPT symmetry is preserved. The general problem of deformed oscillator for non hermitian operators can be applied to the Solid State Physics.

Keywords: Deformed harmonic oscillator; PT symmetry; CPT symmetry; non-Hermitian operators.

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1. Introduction

In a recent paper with the titled "Non-commutative Geometry and Applications in Physics" Jannussis *et al.*[1] have investigated the Bopp–Kubo formulation and the Weyl–Wigner–Moyal symbol calculus for the non-commutative Geometry. The Bopp–Kubo formulation and the symbol calculus are explicitly related to each other.[2]

By using the general case of the coordinates and momentum in three dimensions, which satisfy the commutation relations:

$$\begin{aligned} &[\hat{q}_1, \hat{q}_2] = i\theta\,, \quad [\hat{p}_1, \hat{p}_2] = i\lambda\,, \quad [\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk} \\ &[\hat{q}_2, \hat{q}_3] = [\hat{q}_3, \hat{q}_1] = 0\,, \quad [\hat{p}_2, \hat{p}_3] = [\hat{p}_3, \hat{p}_1] = 0\,. \end{aligned} \tag{1}$$

For the case of two dimensions, in the non-commuting geometry,[3–15] we obtain the operator $\mathcal{H}$ from the simple harmonic oscillator.

$$\mathcal{H} = \frac{\alpha}{2m}\hat{p}_1^2 + \frac{\beta m\omega^2}{2}\hat{q}_1^2 \tag{2}$$

Before studying the real deformed harmonic oscillator, we consider the two cases

$$\lambda = 0\,,\ \theta \neq 0 \quad \text{and} \quad \lambda \neq 0\,,\ \theta = 0 \tag{3}$$

and obtain the following operators in q_j and p_j representations:

$$\mathcal{H}_1 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_1^2} + \frac{m\omega^2}{2}\left(q_1 + \frac{i\theta}{2}\frac{\partial}{\partial q_2}\right)^2 \tag{4}$$

$$\mathcal{H}_2 = \frac{1}{2m}\left(p_1 + i\frac{\lambda}{2}\frac{\partial}{\partial p_2}\right)^2 - \frac{\hbar^2 m\omega^2}{2}\frac{\partial^2}{\partial p_1^2} \tag{5}$$

with the corresponding Shrödinger equations

$$\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_1^2} + E_1 - \frac{m\omega^2}{2}\left(q_1 + i\frac{\theta}{2}\frac{\partial}{\partial q_2}\right)^2\right]\Psi_1(q_1, q_2) = 0 \tag{6}$$

$$\left[\frac{\hbar^2 m\omega^2}{2}\frac{\partial^2}{\partial p_1^2} + E_2 - \frac{1}{2m}\left(p_1 + i\frac{\lambda}{2}\frac{\partial}{\partial p_2}\right)^2\right]\Psi_2(p_1, p_2) = 0\,. \tag{7}$$

The solutions of the above equations take the form

$$\Psi_1(q_1, q_2) = e^{ikq_2}F_1(q_1) \tag{8}$$

$$\Psi_2(p_1, p_2) = e^{i\frac{\nu}{\hbar}p_2}F_2(p_1) \tag{9}$$

where k and ν are real constant parameters.

The functions $F_1(q_1)$ and $F_2(p_1)$ satisfy the equations

$$\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_1^2} + E_1 - \frac{m\omega^2}{2}\left(q_1 - \frac{\theta k}{2}\right)^2\right]F_1(q_1) = 0 \tag{10}$$

$$\left[\frac{\hbar^2 m\omega^2}{2}\frac{\partial^2}{\partial p_1^2} + E_2 - \frac{1}{2m}\left(p_1 - \frac{\nu\lambda}{2\hbar}\right)^2\right]F_2(p_1) = 0 \tag{11}$$

and for

$$q_1 - \frac{\theta k}{2} = x\,, \quad p_1 - \frac{\nu\lambda}{2\hbar} = y \tag{12}$$

we obtain two simple harmonic oscillators i.e.

$$\left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + E_1 - \frac{m\omega^2}{2}x^2\right] F_1(x) = 0 \tag{13}$$

$$\left[\frac{\hbar^2 m\omega^2}{2}\frac{\partial^2}{\partial y^2} + E_2 - \frac{1}{2m}y^2\right] F_2(y) = 0 \tag{14}$$

with the known eigenvalues

$$E_1 = E_2 = \hbar\omega\left(n + \frac{1}{2}\right) \tag{15}$$

which are infinitely degenerate.

The present paper is organized as follows. In Sec. 2 we will study the Hermitian deformed harmonic oscillator i.e. $\alpha = \beta = 1$, m real and the PT symmetry. In Sec. 3 we will study the non-Hermitian operators $\mathcal{H}$ with complex mass, $\alpha = \beta = 1$ and the CPT symmetry. In Sec. 4 we will investigate the operator $\mathcal{H}$ with the complex parameters m, α, β and the CPT symmetry. Section 5 is devoted to concluding remarks.

2. Real Deformed Harmonic Oscillator

For the case

$$\lambda \neq 0 \quad \text{and} \quad \theta \neq 0 \tag{16}$$

in the q_i representation, according to the relation (1), for the simple harmonic oscillator we obtain the deformed harmonic oscillator.

$$\mathcal{H}(p,q) = \frac{1}{2m}\left(p_1 + \frac{\lambda}{2\hbar}q_2\right)^2 + \frac{m\omega^2}{2}\left(q_1 - \frac{\theta}{2\hbar}p_2\right)^2 . \tag{17}$$

The above Hamiltonian can be written

$$\mathcal{H} = \left(\frac{1}{2m}p_1^2 + \frac{1}{2}m\omega^2 q_1^2\right) + k^2\left(\frac{1}{2m}p_2^2 + \frac{1}{2}m\Omega^2 q_2^2\right) - k\omega\left(q_1 p_2 - \frac{\Omega}{\omega}p_1 q_2\right) \tag{18}$$

where

$$\Omega = \frac{\lambda}{m^2\omega\theta}, \quad k = \frac{m\omega\theta}{2\hbar} . \tag{19}$$

In the q_j representation the Hamiltonian (18) takes the form

$$\begin{aligned}\mathcal{H} = &-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_1^2} + k^2\frac{\partial^2}{\partial q_2^2}\right) + ik\omega\hbar\left(q_1\frac{\partial}{\partial q_2} - \frac{\Omega}{\omega}q_2\frac{\partial}{\partial q_1}\right)\\ &+ \frac{m\omega^2}{2}q_1^2 + \frac{m\Omega^2}{2}q_2^2 .\end{aligned} \tag{20}$$

The corresponding Schrödinger equation is

$$\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_1^2}+k^2\frac{\partial^2}{\partial q_2^2}\right)\Psi(q_1,q_2)-ik\omega\hbar\left(q_1\frac{\partial}{\partial q_2}-\frac{\Omega}{\omega}q_2\frac{\partial}{\partial q_1}\right)\Psi(q_1,q_2)$$
$$+\left(E-\frac{m\omega^2}{2}q_1^2-\frac{m\Omega^2}{2}q_2^2\right)\Psi(q_1,q_2)=0\,. \tag{21}$$

For

$$\Psi(q_1,q_2)=\exp\left\{-\frac{m\omega}{2\hbar}q_1^2-\frac{m\Omega}{2\hbar}q_2^2\right\}F(q_1,q_2) \tag{22}$$

the function $F(q_1,q_2)$ satisfies the equation

$$\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_1^2}+k^2\frac{\partial^2}{\partial q_2^2}\right)F(q_1,q_2)-\hbar\omega\left(q_1-ik\frac{\Omega}{\omega}q_2\right)\left(\frac{\partial}{\partial q_1}+ik\frac{\partial}{\partial q_2}\right)F(q_1,q_2)$$
$$+\left(E-\frac{\hbar\omega}{2}-k^2\frac{\hbar\Omega}{2}\right)F(q_1,q_2)=0\,. \tag{23}$$

For

$$F(q_1,q_2)=\frac{1}{2\pi i}\int_c\exp\left\{\left(q_1-ik\frac{\Omega}{\omega}q_2\right)z\right\}\phi(z)dz \tag{24}$$

and after some algebra, we obtain

$$\phi(z)=\exp\left\{-\frac{\hbar}{4m\omega}\left(1-k^2\frac{\Omega}{\omega}\right)z^2\right\}\frac{1}{z^{\frac{E}{\hbar\omega}(1+k^2\frac{\Omega}{\omega})+\frac{1}{2}}} \tag{25}$$

so that

$$F(q_1,q_2)=\frac{1}{2\pi i}\int_c\exp\left\{-\frac{\hbar}{4m\omega}\left(1-k^2\frac{\Omega}{\omega}\right)z^2+\left(q_1-ik\frac{\Omega}{\omega}q_2\right)z\right\}\frac{dz}{z^{\frac{E}{\hbar\omega}(1+\frac{\lambda\theta}{4\hbar^2})+\frac{1}{2}}}\,. \tag{26}$$

According to the integral[17]

$$\frac{H_n(u)}{n!}=\frac{1}{2\pi i}\int^{(0_+)}e^{-s^2+2su}\frac{ds}{s^{n+1}} \tag{27}$$

and for

$$z=\frac{2s}{\sqrt{\frac{\hbar}{m\omega}(1-k^2\frac{\Omega}{\omega})}} \tag{28}$$

the solution (26) takes the form

$$F(q_1,q_2)=\left(\frac{\hbar}{m\omega}\left(1-k^2\frac{\Omega}{\omega}\right)\right)^{n/2}\frac{1}{2^n n!}H_n\left(\frac{q_1-ik\frac{\Omega}{\omega}q_2}{\sqrt{\frac{\hbar}{m\omega}(1-k^2\frac{\Omega}{\omega})}}\right) \tag{29}$$

with the eigenvalues

$$E_n=\hbar\omega\left(1+\frac{\lambda\theta}{4\hbar^2}\right)\left(n+\frac{1}{2}\right) \tag{30}$$

where $H_n(u)$ are the Hermite polynomials. Consequently the eigenfunctions are

$$\Psi(q_1, q_2) = N_n \exp\left\{-\frac{m\omega}{2\hbar}q_1^2 - \frac{m\Omega}{2\hbar}q_2^2\right\} H_n\left(\frac{q_1 - i\mu\sqrt{\frac{\Omega}{\omega}}q_2}{\sqrt{\frac{\hbar}{m\omega}(1-\mu^2)}}\right) \tag{31}$$

where

$$\mu = \sqrt{\frac{\lambda\theta}{2\hbar}} \tag{32}$$

and

$$N_n = \sqrt{\frac{m}{\hbar\pi}}\sqrt[4]{\Omega\omega}\frac{1}{\sqrt{2^n n!}}\left(\frac{1-\mu^2}{1+\mu^2}\right)^{n/2} \tag{33}$$

is the normalization factor.

For

$$q_1 = \sqrt{\frac{\hbar}{m\omega}}x\,, \quad q_2 = \sqrt{\frac{\hbar}{m\Omega}}y \tag{34}$$

the eigenfunctions (31) yield

$$\Psi_n(x, y) = N_n \exp\left\{-\frac{1}{2}(x^2 + y^2)\right\} H_n\left(\frac{x - i\mu y}{\sqrt{1-\mu^2}}\right). \tag{35}$$

which are orthonormal.

From the Hamiltonian (17) we can define the annihilation and the creation operators A, A^+ as

$$A = \frac{1}{\sqrt{2\hbar(1 + \frac{\lambda\theta}{4\hbar^2})}}\left[\sqrt{m\omega}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right) + i\frac{\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2}{\sqrt{m\omega}}\right] \tag{36}$$

$$A^+ = \frac{1}{\sqrt{2\hbar(1 + \frac{\lambda\theta}{4\hbar^2})}}\left[\sqrt{m\omega}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right) - i\frac{\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2}{\sqrt{m\omega}}\right] \tag{37}$$

with the commutation relation

$$[A, A^+] = 1\,. \tag{38}$$

The number operator A^+A satisfies the relation

$$A^+A = \frac{1}{\hbar\omega(1 + \frac{\lambda\theta}{4\hbar^2})}\mathcal{H} - \frac{1}{2} \tag{39}$$

and according to the eigenfunctions (31) and the eigenvalues (30) we obtain

$$A^+A\Psi_n(x, y) = n\Psi_n(x, y) \to A^+A|n\rangle = n|n\rangle\,. \tag{40}$$

The operators (36) and (37) in a_1, a_2, a_1^+, a_2^+ take the form

$$A = \frac{1}{\sqrt{1+\mu^2}}(a_1 + i\mu a_2)\,, \quad A^+ = \frac{1}{\sqrt{1+\mu^2}}(a_1^+ - i\mu a_2^+)\,. \tag{41}$$

From the relations (40) we obtain

$$A^+A = \frac{1}{\sqrt{1+\mu^2}}\left(a_1^+a_1 + \mu^2 a_2^+a_2 - i\mu(a_1a_2^+ - a_1^+a_2)\right) \tag{42}$$

and according to the relation (38) we see that there exists a common basis $|n\rangle$ for

$$a_1^+a_1|n\rangle = n|n\rangle\,, \quad a_2^+a_2|n\rangle = n|n\rangle \tag{43}$$

and

$$A|n\rangle = \frac{1+i\mu}{\sqrt{1+\mu^2}}\sqrt{n}\,|n-1\rangle\,, \quad A^+|n\rangle = \frac{1-i\mu}{\sqrt{1+\mu^2}}\sqrt{n+1}\,|n+1\rangle\,. \tag{44}$$

The coherent states, according to the relation (41) satisfy the equation

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+\mu^2}}\left[\left(x+\frac{\partial}{\partial x}\right) + i\mu\left(y+\frac{\partial}{\partial y}\right)\right]f(x,y) = \frac{\alpha+i\mu\beta}{\sqrt{1+\mu^2}}f(x,y) \tag{45}$$

where α, β complex parameters. For

$$f(x,y) = f(x)f(y) \tag{46}$$

we have

$$\left(x+\frac{\partial}{\partial x}\right)f(x) = \sqrt{2}\alpha f(x)\,, \quad \left(y+\frac{\partial}{\partial y}\right)f(y) = \sqrt{2}\beta f(y) \tag{47}$$

or

$$f(q_1) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left\{-\frac{|\alpha|^2}{2} + \frac{\alpha^2}{2} - \frac{m\omega}{2\hbar}q_1^2 + \sqrt{2}\sqrt{\frac{m\omega}{\hbar}}\alpha q_1\right\} \tag{48}$$

$$f(q_2) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left\{-\frac{|\beta|^2}{2} + \frac{\beta^2}{2} - \frac{m\Omega}{2\hbar}q_2^2 + \sqrt{2}\sqrt{\frac{m\Omega}{\hbar}}\beta q_2\right\}. \tag{49}$$

The mean values of the operators q_1, $-i\hbar\frac{\partial}{\partial q_1}$, q_2, $-i\hbar\frac{\partial}{\partial q_2}$, are very simple to calculate and the dispersion relations are the following:

$$\begin{aligned}(\Delta q_1)^2 = \tfrac{\hbar}{2m\omega}\,, \quad (\Delta p_1)^2 = \tfrac{\hbar m\omega}{2}\\ (\Delta q_2)^2 = \tfrac{\hbar}{2m\Omega}\,, \quad (\Delta p_2)^2 = \tfrac{\hbar m\Omega}{2}\,.\end{aligned} \tag{50}$$

From the above relations, we obtain the minimum uncertainty relations

$$(\Delta q_1)(\Delta p_1) = \frac{\hbar}{2}\,, \quad (\Delta q_2)(\Delta p_2) = \frac{\hbar}{2}\,. \tag{51}$$

The time evolution of the operators $\hat{q}_j$, $\hat{p}_j$ for the Hamiltonian operators (17) is obtained by the Heisenberg picture with the equations of motion

$$\frac{d\hat{q}_1}{dt} = \frac{1}{m}\left(\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2\right), \quad \frac{d\hat{p}_1}{dt} = -m\omega^2\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right) \tag{52}$$

$$\frac{d\hat{q}_2}{dt} = -\frac{m\omega^2\theta}{2\hbar}\left(\hat{p}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right), \quad \frac{d\hat{p}_2}{dt} = -\frac{\lambda}{2\hbar m}\left(\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2\right). \tag{53}$$

From the above equations we obtain

$$\frac{d\hat{p}_2}{dt} = -\frac{\lambda}{2\hbar}\frac{d\hat{q}_1}{dt} \to \hat{p}_2(t) = -\frac{\lambda}{2\hbar}\hat{q}_1 + C_1 \tag{54}$$

$$\frac{d\hat{q}_2}{dt} = \frac{\theta}{2\hbar}\frac{d\hat{p}_1}{dt} \to \hat{q}_2(t) = \frac{\theta}{2\hbar}\hat{p}_1 + C_2 \tag{55}$$

and for $t = 0$ we take the expressions

$$\hat{p}_2(t) = \hat{p}_2(0) - \frac{\lambda}{2\hbar}\left[\hat{q}_1(t) - \hat{q}_1(0)\right] \tag{56}$$

$$\hat{q}_2(t) = \hat{q}_2(0) + \frac{\theta}{2\hbar}\left[\hat{p}_1(t) - \hat{p}_1(0)\right]. \tag{57}$$

Substituting Eqs. (54) and (55) into Eq. (52) we obtain

$$\frac{d\hat{q}_1(t)}{dt} = \frac{1}{m}\left[\left(1 + \frac{\lambda\theta}{4\hbar^2}\right)\hat{p}_1(t) + \frac{\lambda}{2\hbar}\hat{q}_2(0) - \frac{\lambda\theta}{4\hbar^2}\hat{p}_1(0)\right] \tag{58}$$

$$\frac{d\hat{p}_1(t)}{dt} = -m\omega\left[\left(1 + \frac{\lambda\theta}{4\hbar^2}\right)\hat{q}_1(t) - \frac{\theta}{2\hbar}\hat{p}_2(0) - \frac{\lambda\theta}{4\hbar^2}\hat{q}_1(0)\right]. \tag{59}$$

The solutions of the above equations are

$$\begin{aligned}\hat{q}_1(t) = &\left(\frac{\lambda\theta}{4\hbar^2} + \cos(\tilde{\omega}t)\right)\frac{\hat{q}_1(0)}{1 + \frac{\lambda\theta}{4\hbar}} + \frac{\theta(1 - \cos(\tilde{\omega}t))}{2\hbar(1 + \frac{\lambda\theta}{4\hbar^2})}\hat{p}_2(0) \\ &+ \frac{1}{m\tilde{\omega}}\left(\hat{p}_1(0) + \frac{\lambda}{2\hbar}\hat{q}_2(0)\right)\sin(\tilde{\omega}t)\end{aligned} \tag{60}$$

$$\begin{aligned}\hat{p}_1(t) = \frac{1}{1 + \frac{\lambda\theta}{4\hbar^2}}&\left[\left(\frac{\lambda\theta}{4\hbar^2} + \cos(\tilde{\omega}t)\right)\hat{p}_1(0) - \frac{\lambda}{2\hbar}(1 - \cos(\tilde{\omega}t))\hat{q}_2(0)\right. \\ &\left. - \frac{m\tilde{\omega}}{1 + \frac{\lambda\theta}{4\hbar^2}}\left(\hat{q}_1(0) - \frac{\theta}{2\hbar}\hat{p}_2(0)\right)\sin(\tilde{\omega}t)\right]\end{aligned} \tag{61}$$

where

$$\tilde{\omega} = \omega\left(1 + \frac{\lambda\theta}{4\hbar^2}\right). \tag{62}$$

From the above solutions we obtain

$$[\hat{q}_1(t), \hat{p}_1(t)] = [\hat{q}_1(0), \hat{p}_1(0)] = i\hbar \tag{63}$$

and according to the relations (55), (56) we obtain the commutator

$$[\hat{q}_2(t), \hat{p}_2(t)] = [\hat{q}_2(0), \hat{p}_2(0)] = i\hbar\,. \tag{64}$$

For the conservation of PT symmetry, the following relations hold

$$P : q_i \to -q_i, p_j \to -p_j\,, \quad T :\ p_j \to p_j, i \to -i\,. \tag{65}$$

From the above relations and the real spectrum (30) we see that PT symmetry is preserved.

3. Deformed Harmonic Oscillator with Complex Mass

In this section we consider the non-Hermitian deformed oscillator

$$\mathcal{H}_1 = \frac{1}{2m}\left(\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_1\right)^2 + \frac{m\omega^2}{2}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_1\right)^2 \tag{66}$$

with m complex and $\mathrm{Re}(m) > 0$.

The operators $\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_1$ and $\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_1$ satisfy the following commutation relation:

$$\left[\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_1, \hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_1\right] = i\hbar\left(1 + \frac{\lambda\theta}{4\hbar^2}\right) = i\hbar' \tag{67}$$

where $i\hbar' = i\hbar(1 + \frac{\lambda\theta}{4\hbar^2})$ describes the deformation.

For the case of complex mass the PT symmetry is no more symmetric but CPT symmetric, i.e.

$$C : i \to -i; \quad P : q_j \to -q_j, p_j \to -p_j; \quad T : p_j \to p_j, i \to -i \tag{68}$$

and according to real spectrum (30) the CPT symmetry is preserved.

Non-Hermitian Hamiltonians with real spectrum have been recently studied and exploited in an intensive way by Bender *et al.*,[18–22] leading to new trends in fundamental quantum mechanics. Also Bender *et al.*[23] exhibit the CPT symmetry and Jannussis *et al.*[24,25] have studied the non-unitary squeeze operators with two complex parameter on non-Hermitian harmonic oscillator.

Before we start studying the general case, we shall refer to the problem of complex mass and frequency for the simple harmonic oscillator, where the CPT symmetry holds and the eigenvalues spectrum is discrete and complex, so that the CPT symmetry is spontaneously broken.[25]

In the next section we shall study the general case of the non-Hermitian deformed harmonic oscillator.

4. Non-Hermitian Deformed Harmonic Oscillator

In this section we study the operator

$$\mathcal{H} = \frac{\alpha}{2m}\left(\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2\right)^2 + \frac{\beta m\omega^2}{2}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right)^2 . \tag{69}$$

For $m = \alpha M$ the above operator takes the form

$$\mathcal{H} = \frac{1}{2M}\left(\hat{p}_1 + \frac{\lambda}{2\hbar}\hat{q}_2\right)^2 + \frac{M\Omega^2}{2}\left(\hat{q}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right)^2 \tag{70}$$

where

$$\Omega^2 = \alpha\beta\omega^2 , \quad M_1 = \frac{m_1\alpha_1 + m_2\alpha_2}{\alpha_1^2 + \alpha_2^2} , \quad M_2 = \frac{m_2\alpha_1 - m_1\alpha_2}{\alpha_1^2 + \alpha_2^2} \tag{71}$$

and

$$\alpha\beta = (\alpha_1\beta_1 - \alpha_2\beta_2) + i(\alpha_2\beta_1 + \alpha_1\beta_2) . \tag{72}$$

Consequently, the study of the operator (69) is reduced to the problem of complex mass and frequency, i.e. $\Omega = \Omega_1 + i\Omega_2$ with the discrete complex spectrum

$$E_n = \hbar(\Omega_1 + i\Omega_2)\left(1 + \frac{\lambda\theta}{4\hbar^2}\right)(n + 1/2) \tag{73}$$

and

$$E_n^* = \hbar(\Omega_1 - i\Omega_2)\left(1 + \frac{\lambda\theta}{4\hbar^2}\right)(n + 1/2) \tag{74}$$

are the eigenvalues of the adjoint operator $\mathcal{H}^+$. In the above case we have spontaneous breaking of the CPT symmetry.

For the case

$$\alpha_1\beta_2 + \alpha_2\beta_1 = 0 \to \beta_2 = -\frac{\alpha_1}{\alpha_2\beta_1} \tag{75}$$

we obtain

$$\Omega^2 = \omega^2 \frac{\beta_1}{\alpha_1}|\alpha|^2 \quad \text{or} \quad \Omega = \omega\sqrt{\frac{\beta_1}{\alpha_1}}|\alpha| \tag{76}$$

and the eigenvalues are real, discrete and positive namely

$$E_n = \hbar\omega\sqrt{\frac{\beta_1}{\alpha_1}}|\alpha|\left(1 + \frac{\lambda\theta}{4\hbar^2}\right)(n + 1/2)\,. \tag{77}$$

This final result preserves the CPT symmetry. For further research the main problem is the connection between non-commutative geometry for non-Hermitian operators as the relation and CPT symmetries.

5. Conclusion

In the present paper we have examined the non-Hermitian deformed harmonic oscillator $\mathcal{H}$ and the behavior of the PT and CPT symmetries, and we have obtained the following results:

For the case of the Hermitian deformed harmonic oscillator we have found the eigenvalues (30) and the corresponding eigenfunctions (31), the annihilation and creation operators A, A^+ with the common basis $|n\rangle$ and the presentation of the PT symmetry. Also, for the case of the deformed harmonic oscillator with complex mass, the PT symmetry is broken, but, according to the real spectrum (30), the CPT symmetry is preserved. Finally, for the general case of the non-Hermitian operator $\mathcal{H}$, the CPT symmetry, from the relations (73) and (74), is spontaneously broken. According to the relation (75), since the spectrum is real discrete and positive, the CPT symmetry is preserved.

In Refs. 26 (Displaced squeezed number states of the phonon field in polar semiconductors) and 27 (PT and CPT symmetries for non-Hermitian operators), we have enlarged the Max Planck law for complex frequency $E = \hbar(\omega_1 + i\omega_2)$.

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