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FERMI–DIRAC STATISTICS FOR FREE ELECTRONS IN UNIFORM ELECTRIC AND MAGNETIC FIELDS

A. D. JANNUSSIS, A. STREKLAS and K. VLACHOS

Department of Theoretical Physics, University of Patras, Patras, Greece

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In this paper we study the De Haas–Van Alphen effect when an electric field is present. We prove that for sufficiently weak electric fields, where the conditions are favorable for energy quantization, the free energy is a quasi-periodical function with respect to the fields. As a consequence we find, for the magnetic susceptibility, periodical expressions which are easily reduced to the ones known for the De Haas–Van Alphen effect when the electric field vanishes.

1. Introduction

The partition function of free electrons in uniform magnetic and electric fields has been calculated by Jannussis[1]) with the help of the density matrix method. Quite recently the same partition function as well as the density matrix have been calculated[2]) using standard techniques of operator-ordering.

We assume that the magnetic field is parallel to the z-direction namely $\boldsymbol{H} = (0, 0, H)$, and that the electric field is parallel to an arbitrary direction with components E_1, E_2, E_3. The potentials have been chosen to be $\boldsymbol{A} = \frac{1}{2}\boldsymbol{H} \times \boldsymbol{r}$ and $\varphi = -\boldsymbol{E} \cdot \boldsymbol{r}$, then the density matrix is

$$\begin{aligned}\rho(r, r', b) = {} & \left(\frac{m}{2\pi\hbar^2 b}\right)^{3/2} \frac{\mu H b}{\sinh \mu H b}\, e^{f(b)+e(\boldsymbol{E}\cdot \boldsymbol{r}')b} \\ & \times \exp\biggl(-\frac{m}{2b\hbar^2}\{2\mu H b\, \mathrm{i}(x'y - xy') + (z-z')^2 \\ & + \mu H b \coth \mu H b[(x-x')^2 - (y-y')^2]\} + e\,\frac{E_3 b}{2}(z-z') \\ & + e\left\{\frac{E_1 b}{2} - \mathrm{i}\,\frac{E_2 b}{2\mu H b}(1 - \mu H b \coth \mu H b\right\}(x-x') \\ & + e\left\{\frac{E_2}{2}b + \mathrm{i}\,\frac{E_1 b}{2\mu H b}(1 - \mu H b \coth \mu H b)\right\}(y-y')\biggr), \end{aligned} \tag{1.1}$$

where

$$f(b) = \frac{\hbar^2 e^2}{24m} E_3^2 b^3 - \frac{\hbar^2 e^2 (E_1^2 + E_2^2)}{8m(\mu H)^3}\{\mu H b - (\mu H b)^2 \coth \mu H b\}.$$

We can now easily obtain the partition function using the relation

$$Z(b) = \int \rho(\boldsymbol{r}, \boldsymbol{r}, b)\, \mathrm{d}\boldsymbol{r}. \tag{1.2}$$

Because of the term $\exp(e(\boldsymbol{E} \cdot \boldsymbol{r}')b)$, which represents the contribution of the electric field to the density matrix, it is necessary to assume that the particle is enclosed in a box of edge-length L. Periodic boundary conditions have now to be taken into account so that the limits of integration are restricted in the interval $[-\frac{1}{2}L_i, \frac{1}{2}L_i]$.

To avoid the divergence of this density matrix we have to work in an inertial frame of reference S' which moves with constant velocity $\boldsymbol{v} = 2c(\boldsymbol{E} \times \boldsymbol{H})/H^2$. In this frame the observer understands only the rotational motion of the particle and consequently only the magnetic field seems to influence the motion[3]).

The scalar potential becomes zero:

$$\varphi' = \varphi - \frac{1}{c}\boldsymbol{v} \cdot \mathbf{A} = 0. \tag{1.3}$$

In order to obtain the new hamiltonian in the S' system, namely

$$\mathcal{H}' = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\mathbf{A}'(\boldsymbol{r}, t)\right)^2 + e\varphi'(\boldsymbol{r}, t), \tag{1.4}$$

We perform the following gauge transformations:

$$\mathbf{A}' = \mathbf{A} + \nabla\Phi, \qquad \varphi' = \varphi - \frac{1}{c}\frac{\partial}{\partial t}\Phi, \tag{1.5}$$

where

$$\Phi(\boldsymbol{r}, t) = -ct\boldsymbol{E} \cdot \boldsymbol{r}. \tag{1.6}$$

We now consider the primed hamiltonian

$$\mathcal{H}'(b) = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\mathbf{A}'(\boldsymbol{r}, -\mathrm{i}\hbar b)\right)^2 + e\varphi'(\boldsymbol{r}, -\mathrm{i}\hbar b), \tag{1.7}$$

where

$$\mathbf{A}' = \tfrac{1}{2}\boldsymbol{H} \times \boldsymbol{r} + \mathrm{i}\hbar c b\boldsymbol{E}, \qquad \varphi' = 0. \tag{1.8}$$

It is known that if $\rho(\boldsymbol{r}, \boldsymbol{r}', b)$ is a solution of the Bloch equation

$$\mathcal{H}\rho(\boldsymbol{r}, \boldsymbol{r}', b) = -\frac{\partial}{\partial b}\rho(\boldsymbol{r}, \boldsymbol{r}', b), \qquad \rho(\boldsymbol{r}, \boldsymbol{r}', 0) = \delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{1.9}$$

then the density matrix

$$\rho'(\boldsymbol{r}, \boldsymbol{r}', b) = \exp\left(\mathrm{i}\frac{e}{\hbar c}\Phi(\boldsymbol{r}, -\mathrm{i}\,\hbar b)\right)\rho(\boldsymbol{r}, \boldsymbol{r}', b) \tag{1.10}$$

$$= \exp(-e\boldsymbol{E}\cdot \boldsymbol{r}b)\rho'(\boldsymbol{r}, \boldsymbol{r}', b)$$

is a solution of the primed Bloch equation:

$$\mathcal{H}'(b)\rho'(\boldsymbol{r}, \boldsymbol{r}', b) = -\frac{\partial}{\partial b}\rho'(\boldsymbol{r}, \boldsymbol{r}', b), \qquad \rho'(\boldsymbol{r}, \boldsymbol{r}', 0) = \delta(\boldsymbol{r} - \boldsymbol{r}). \tag{1.11}$$

It is clear that the chosen gauge eliminates the divergent part of the density matrix.

To obtain the partition function per unit volume we put $\boldsymbol{r} = \boldsymbol{r}'$ and we have

$$Z(b) = \int \rho'(\boldsymbol{r}, \boldsymbol{r}', b)\,\mathrm{d}\boldsymbol{r} = \left(\frac{m}{2\pi\hbar^2 b}\right)^{3/2}\frac{\mu H b}{\sinh(\mu H b)}\exp f(b). \tag{1.12}$$

Expanding the coth-function in powers up to the second term we find the following approximation of the partition function.

$$Z(b) = \left(\frac{m}{2\pi\hbar^2 b}\right)^{3/2}\frac{\mu H b}{\operatorname{sh}(\mu H b)}\exp\left(\frac{\hbar^2 e^2}{24m}E^2 b^3\right), \tag{1.13}$$

which holds for sufficiently weak magnetic field.

We see that although the gauge transformations in theory do not affect the fields the appropriate boundary conditions in the two cases are quite different. This fact has also been pointed out for the wave function in purely electric field in ref. 4.

As can be easily seen, the partition function (1.12) is a product of the electric field and the magnetic field partition functions and of another term which expresses the interaction of the two fields. The E_3 component in the last exponential does not contribute very much and this can be justified by the fact that the two fields do not actually interact. On the contrary, the other components of the electric field do contribute and cause many complications in the calculations, since the single poles of the magnetic field become singularities because of the term $\coth(\mu H b)$; a direct consequence of the degeneracy of the energy eigenvalues.

2. Boltzmann statistics

Within the framework of the validity of Boltzmann statistics, the free energy and the magnetic susceptibility are given by the following relations:

$$F = -NkT \log Z(b), \tag{2.1}$$

$$\chi = N\frac{kT}{\mu H}\frac{\partial}{\partial H}\log Z(b). \tag{2.2}$$

A direct calculation yields

$$\chi = \chi_0 - N\,\frac{\hbar^2 e^2\mu^2(E_1^2+E_2^2)}{8m(kT)^4}\left(-\frac{2}{\Psi^4}+\frac{\coth\Psi}{\Psi^3}+\frac{\coth^2\Psi-1}{\Psi^2}\right), \tag{2.3}$$

$$\Psi = \mu H/kT,$$

where χ_0 is the susceptibility of electrons in a magnetic field. The constant susceptibility is given by the following relation:

$$\chi_0 = -\frac{1}{3}\frac{N\mu^2}{kT}\left(1+\frac{\hbar^2 e^2(E_1^2+E_2^2)}{60m(kT)^3}\right), \tag{2.4}$$

which shows the resulting increase, independent of the component E_3. In the sequel, instead of the Boltzmann statistics which has been studied in ref. 1 we will use Fermi–Dirac statistics, where the E_3 component has been eliminated.

3. Fermi–Dirac statistics

In low temperatures we must employ Fermi–Dirac statistics. We must therefore calculate the free energy per unit volume from the relation

$$F - n\zeta = -2kT\sum_i \log(1+\mathrm{e}^{(\zeta-\epsilon_i)/kT}). \tag{3.1}$$

The function F can be obtained from the classical partition function $Z(b)$ by using the Laplace transforms in the following way. We define an auxiliary function $z(\epsilon)$ by the equation [3])

$$z(\epsilon) = \frac{1}{2\pi\,\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{\epsilon b}\,\frac{Z(b)}{b^2}\,\mathrm{d}b. \tag{3.2}$$

The contour of integration is parallel to the imaginary axis and the constant c must be chosen so that all the singularities of the integrand are on the left. By the help of $z(\epsilon)$ we easily obtain the free energy through the relation

$$F - n\zeta = 2\int_0^\infty z(\epsilon)\,\frac{\partial f}{\partial \epsilon}\,\mathrm{d}\epsilon, \tag{3.3}$$

where $f(\epsilon)$ is the Fermi–Dirac distribution function

$$f(\epsilon) = \frac{1}{e^{(\epsilon-\zeta)/kT} + 1}. \tag{3.4}$$

At low temperatures $\partial f/\partial\epsilon$ is almost a delta function, so that the essential part of the problem is just the calculation of $z(\epsilon)$. In order to calculate the integral (3.2) it is convenient to change the path of integration into the well-known Bromwich contour. So we need to consider only the contribution from the residues and from the path parallel to the negative real axis. That is

$$z(\epsilon) = \frac{1}{2\pi i} \int_{-\infty}^{0_+} e^{\epsilon b} \frac{Z(b)}{b^2} db + \sum_{r=1}^{\infty} \{R(r\pi\,i) + R(-r\pi\,i)\}. \tag{3.5}$$

The contour integral in (3.5) can be calculated as follows: we write

$$\frac{1}{2\pi i} \int_{-\infty}^{0_+} e^{\epsilon b} \frac{Z(b)}{b^2} db = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \frac{1}{2\pi i} \int_{-\infty}^{0_+} \left[\frac{1}{b^{7/2}} - \frac{(\mu H)^2}{6b^{3/2}} + \frac{\hbar^2 e^2(E_1^2 + E_2^2)}{24mb^{1/2}}\right] e^{\epsilon b}\, db$$

$$- \int_{-\infty}^{0_+} \left[\frac{1}{b^{7/2}} - \frac{(\mu H)^2}{6b^{3/2}} + \frac{\hbar^2 e^2(E_1^2 + E_2^2)}{24mb^{1/2}} - \frac{Z(b)}{b^2}\right] e^{\epsilon b}\, db\,. \tag{3.6}$$

The first integral can be evaluated by means of Hankel's formula:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{0_+} t^{-z}\, e^t\, dt\,, \tag{3.7}$$

while the second integral can be transformed into a real infinite integral, since the integrand is of order $b^{1/2}$ near the origin. So we obtain

$$\frac{1}{2\pi i} \int_{-\infty}^{0_+} e^{\epsilon b} \frac{Z(b)}{b^2} db = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \left\{\frac{8}{18\sqrt{\pi}} \epsilon^{5/2} - \frac{(\mu H)^2}{3\sqrt{\pi}} \epsilon^{1/2} + \frac{\hbar^2 e^2(E_1^2 + E_2^2)}{24m\sqrt{\pi}} \epsilon^{-1/2}\right.$$

$$\left. + \frac{1}{\pi} (\mu H)^{5/2} \int_0^{\infty} \left[\frac{1}{y^{7/2}} - \frac{1}{6y^{3/2}} + \frac{\hbar^2 e^2(E_1^2 + E_2^2)}{24m(\mu H)^3 y^{1/2}} - \frac{Z(y)}{y^2}\right] e^{y\epsilon/\mu H}\, dy\right\}. \tag{3.8}$$

To find now the contribution from the residues $\pm r\pi\,i$, we consider the formulas

$$\coth \mu Hb = \frac{1}{\mu Hb} + \sum_{n=1}^{\infty} \frac{2\mu Hb}{(\mu Hb)^2 + r^2\pi^2},$$

$$\frac{\mu Hb}{\sinh(\mu Hb)} = \prod_{n=1}^{\infty} \frac{r^2\pi^2}{(\mu Hb)^2 + r^2\pi^2}. \tag{3.9}$$

If we set

$$\alpha = \hbar^2 e^2 (E_1^2 + E_2^2)/8m(\mu H)^2 , \tag{3.10}$$

we have

$$\begin{aligned}
&\frac{\mu Hb}{\sinh(\mu Hb)} \exp\Big(-\frac{\alpha}{\mu H} \{\mu Hb - (\mu Hb)^2 \coth(\mu Hb)\}\Big) \\
&= \prod_{r=1}^{\infty} \frac{r^2\pi^2}{(\mu Hb)^2 + r^2\pi^2} \exp\Big\{ \frac{2\alpha b(\mu Hb)^2}{(\mu Hb)^2 + r^2\pi^2} \Big\} \\
&= \prod_{r=1}^{\infty} \frac{r^2\pi^2}{(\mu Hb + r\pi\, \mathrm{i})(\mu Hb - r\pi\, \mathrm{i})} \exp\Big\{ \frac{\alpha}{\mu H} \Big(\frac{(\mu Hb)^2}{\mu Hb + r\pi\, \mathrm{i}} + \frac{(\mu Hb)^2}{\mu Hb - r\pi\, \mathrm{i}} \Big) \Big\} ,
\end{aligned} \tag{3.11}$$

from which it is clear that we can isolate the singularity $r\pi\,\mathrm{i}$ if we write

$$\begin{aligned}
\mathrm{e}^{\epsilon b} \frac{Z(b)}{b^2} &= \Big(\frac{m}{2\pi\hbar^2}\Big)^{3/2} \frac{\mu Hb - r\pi\,\mathrm{i}}{b^{5/2}\sinh(\mu Hb)} \exp\Big(\frac{\alpha}{\mu H} (\mu Hb)^2 \coth(\mu Hb) - \frac{1}{\mu Hb - r\pi\,\mathrm{i}}\Big) \\
&\times \exp\Big(\frac{\epsilon + \alpha}{\mu H}\, r\pi\mathrm{i}\Big) \frac{\mu H}{\mu Hb - r\pi\,\mathrm{i}} \exp\Big\{\frac{\epsilon}{\mu H} (\mu Hb - r\pi\,\mathrm{i}) - \frac{(\alpha/\mu H) r^2\pi^2}{\mu Hb - r\pi\,\mathrm{i}}\Big\} .
\end{aligned} \tag{3.12}$$

The last term contains the singularity and can be expanded through the

$$\begin{aligned}
&\frac{1}{(\mu Hb - r\pi\,\mathrm{i})} \exp\Big(\frac{\epsilon}{\mu H} (\mu Hb - r\pi\,\mathrm{i}) - \frac{(\alpha/\mu H) r^2\pi^2}{\mu Hb - r\pi\,\mathrm{i}}\Big) \\
&= \sum_{k=-\infty}^{\infty} \frac{(\sqrt{(\alpha/\epsilon)}\, r\pi)^k}{(\mu Hb - r\pi\,\mathrm{i})^{k+1}} J_K\Big(\frac{2r\pi}{\mu H} \sqrt{\epsilon\alpha}\Big) .
\end{aligned} \tag{3.13}$$

The remaining part of (3.12) is an analytic function around the point $\mu Hb - r\pi\,\mathrm{i}$.

We assume that the electric field is weak with respect to the magnetic field so we consider the following approximate case:

$$\sqrt{\alpha/\epsilon} \ll 1 . \tag{3.14}$$

With the approximation (3.14) valid we keep only the first term of the expansion (3.13), provided that the Bessel functions are bounded. Consequently, the contribution from the residue $r\pi\,\mathrm{i}$ is

$$R(r\pi\,\mathrm{i}) = -(\mu H)^{5/2} \frac{(-1)^r}{(r\pi)^{5/2}} J_0\Big(\frac{2r\pi}{\mu H} \sqrt{\alpha\epsilon}\Big) \exp\Big(\frac{\epsilon + \alpha}{\mu H}\, r\pi\,\mathrm{i} - \frac{\pi}{4}\mathrm{i}\Big) . \tag{3.15}$$

Collecting now all the terms we find for the function $z(\epsilon)$ the expression

$$\begin{aligned}
z(\epsilon) &= \Big(\frac{m}{2\pi\hbar^2}\Big)^{3/2} \Big\{ \frac{8}{15\sqrt{\pi}} \epsilon^{5/2} - \frac{(\mu H)^2}{3\sqrt{\pi}} \epsilon^{1/2} + \frac{\hbar^2 e^2 (E_1^2 + E_2^2)}{24m\sqrt{\pi}} \epsilon^{-1/2} - 2(\mu H)^{5/2} \\
&\times \sum_{r=1}^{\infty} \frac{(-1)^r}{(r\pi)^{5/2}} J_0\Big(\frac{2r\pi}{\mu H} \sqrt{\epsilon\alpha}\Big) \cos\Big(\frac{\epsilon + \alpha}{\mu H}\, r\pi - \tfrac{1}{4}\pi\Big)\Big\}
\end{aligned} \tag{3.16}$$

the energy levels are given by the relation

$$\epsilon = 2\mu H(n+\tfrac{1}{2}) + \tfrac{1}{2}mc^2(E_1^2 + E_2^2)/H^2 - eE_1x_1 - eE_2x_2. \tag{3.17}$$

So the considered approximation $\sqrt{\alpha/\epsilon} \ll 1$ means that

$$2\mu H(n+\tfrac{1}{2}) \gg eE_1x_1 + eE_2x_2, \tag{3.18}$$

that is the discrete part of the energy is much greater than the continuous one and the condition is favorable for energy quantization.

The calculation proceeds with the determination of the free energy through the relation (3.3). At low temperatures ($kT \ll \zeta$) we may, with respect to the first three terms replace $\partial f/\partial\epsilon$ by $-\delta(\epsilon - \zeta)$; in any case the exact calculation of the normal diamagnetism is out of the domain of this paper. The oscillatory term, however, must be handled with greater care. Set

$$\lambda = \alpha r\pi/\mu H - \tfrac{1}{4}\pi$$

and write

$$J_0\left(\frac{2r\pi}{\mu H}\sqrt{\epsilon\alpha}\right)\cos\left(\frac{\epsilon r\pi}{\mu H} + \lambda\right)$$

$$= R_{t^0}\exp\left\{\frac{\epsilon r\pi}{\mu H}\,\mathrm{i} + \lambda\,\mathrm{i} + \frac{\epsilon r\pi}{\mu H}\,t\,\mathrm{i} + \frac{\alpha r\pi}{\mu H t}\,\mathrm{i}\right\}$$

$$= R_{t^0}\exp\left\{\frac{\zeta r\pi}{\mu H}\,\mathrm{i} + \lambda\,\mathrm{i} + \frac{\zeta r\pi}{\mu H}\,t\,\mathrm{i} + \frac{\alpha r\pi}{\mu H t}\,\mathrm{i}\right\}\exp\left\{\frac{(\epsilon-\zeta)r\pi\,\mathrm{i}}{\mu H}(1+t)\right\}, \tag{3.19}$$

where R_{t^0} denotes the real part of the constant term, and consider the integral

$$\int_0^\infty e^{r\pi(1+t)\,\mathrm{i}(\epsilon-\zeta)/\mu H}\,\frac{\partial f_0}{\partial\epsilon}\,\mathrm{d}\epsilon\,.$$

Put $\eta = (\epsilon - \zeta)/kT$ and replace the lower limit of integration $-\zeta/kT$ by $-\infty$ the error being $O(\exp(-\zeta/kT))$; the integral is then

$$-\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{r\pi(1+t)\,\mathrm{i}kT\eta/\mu H}}{\cosh^2(\eta/2)}\,\mathrm{d}\eta,$$

which is a tabulated one; its value being

$$-\frac{r\pi^2(1+t)kT/\mu H}{\sinh(r\pi^2(1+t)kT/\mu H)}.$$

Here we briefly discuss the considered approximation. It is well known that

$$\int_0^\infty J_0\left(\frac{2r\pi}{\mu H}\sqrt{\epsilon\alpha}\right)\cos\left(\frac{\epsilon r\pi}{\mu H}+\lambda\right)\frac{\partial f}{\partial\epsilon}\,\mathrm{d}\epsilon$$

$$= -R_{t^0}\exp\left\{\frac{\zeta r\pi}{\mu H}\,\mathrm{i}+\lambda\,\mathrm{i}+\frac{\zeta r\pi}{\mu H}\,t\,\mathrm{i}+\frac{\alpha r\pi}{\mu H t}\,\mathrm{i}\right\}\frac{r\pi^2(1+t)kT/\mu H}{\sinh(r\pi^2(1+t)kT/\mu H)}. \tag{3.20}$$

We now expand the expression (3.20) with respect to t and using the well-known properties of the Bessel functions we finally find the following formula for the integral:

$$\sum_{n=0}^{\infty}(-1)^n J_{2n}\left(\frac{2r\pi}{\mu H}\sqrt{\alpha\zeta}\right)\left(\sqrt{\frac{\alpha}{\zeta}}\right)^{2n}C_{2n}\cos\left(\frac{\zeta r\pi}{\mu H}+\lambda\right)$$

$$-\sum_{n=0}^{\infty}(-1)^n J_{2n+1}\left(\frac{2r\pi}{\mu H}\sqrt{\alpha\zeta}\right)\left(\sqrt{\frac{\alpha}{\zeta}}\right)^{2n+1}C_{2n+1}\sin\left(\frac{\zeta r\pi}{\mu H}+\lambda\right), \tag{3.21}$$

where the expansion coefficients C_n are given by the relation

$$C_n = \frac{1}{n!}\left(r\pi^2\frac{kT}{\mu H}\right)^n\frac{\partial^n}{\partial x^n}\frac{x}{\sinh x}\bigg|_{x=r\pi^2 kT/\mu H}. \tag{3.22}$$

Finally, collecting all the terms we find the free energy:

$$F-n\zeta = -2\left(\frac{m}{2\pi\hbar^2}\right)^{3/2}\left\{\frac{8}{15\sqrt{\pi}}\zeta^{5/2}-\frac{(\mu H)^2}{3\sqrt{\pi}}\zeta^{1/2}+\frac{\hbar^2 e^2(E_1^2+E_2^2)}{24m\sqrt{\pi}\zeta^{1/2}}-2(\mu H)^{5/2}\right.$$

$$\left.\times\sum_{r=1}^{\infty}\left\{\frac{(-1)^r}{(r\pi)^{5/2}}\sum_{n=0}^{\infty}(-1)^n J_n\left(\frac{2r\pi}{\mu H}\sqrt{\alpha\zeta}\right)\left(\sqrt{\frac{\alpha}{\zeta}}\right)^n C_n\cos\left(\frac{\zeta+\alpha}{\mu H}r\pi+(n-\tfrac{1}{4})\pi\right)\right\}\right\}, \tag{3.23}$$

which can be easily treated by a computer machine. We note that series (3.23) is rapidly convergent with respect to r, with the possible exception right on the turning point $\sqrt{\zeta}=\sqrt{\alpha}$.

4. Stable states

We now examine the free energy (3.23) in two approximate cases. If we assume that the Fermi energy has the magnitude of a typical atomic binding energy the two conditions $\sqrt{\alpha/\zeta}\ll 1$ and $\sqrt{\alpha/\zeta}\gg 1$ characterize different regimes of particle motion. When $\sqrt{\alpha/\zeta}\ll 1$ there is no possibility of decay and small values of $\sqrt{\alpha/\zeta}$ correspond to more stable states[6]). In addition, the conditions are favorable for energy quantization and the energy gaps are smaller than the Fermi energy. So the quasi-classical conclusions hold and we must expect oscillatory effects[7]).

We keep only the first term of (3.23). Setting

$$F_0(\zeta) = \frac{8}{15\sqrt{\pi}}\zeta^{5/2} - \frac{(\mu H)^2}{3\sqrt{\pi}}\zeta^{1/2} + \frac{\hbar^2 e^2(E_1^2 + E_2^2)}{24m\sqrt{\pi}\,\zeta^{1/2}},$$

we obtain

$$F - n\zeta = -2\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} F_0(\zeta) + \frac{\sqrt{2}m^{3/2}}{\pi^2\hbar^3} kT(\mu H)^{3/2} \times \sum_{r=1}^{\infty} J_0\left(\frac{2r\pi}{\mu H}\sqrt{\zeta\alpha}\right)\cos\left(\frac{\zeta+\alpha}{\mu H} r\pi - \tfrac{1}{4}\pi\right)\frac{(-1)^r}{r^{3/2}\sinh(r\pi^2 kT/\mu H)}. \quad (4.1)$$

Observe that for the limiting case where $E_i \to 0$ we find the well-known free energy of the magnetic field in low temperatures, namely

$$F - n\zeta = -2\left(\frac{m}{2\pi\hbar^2}\right)^{3/2}\left(\frac{8}{15\sqrt{\pi}}\zeta^{5/2} - \frac{(\mu H)^2}{3\sqrt{\pi}}\zeta^{1/2}\right) + \frac{\sqrt{2}\,m^{3/2}}{\pi^2\hbar^3} kT(\mu H)^{3/2} \times \sum_{r=1}^{\infty}\cos\left(\frac{\zeta r\pi}{\mu H} - \tfrac{1}{4}\pi\right)\frac{(-1)^r}{r^{3/2}\sinh(r\pi^2 kT/\mu H)}. \quad (4.2)$$

We see that the crossed component of the electric field offers a new oscillating term coming through a zeroth-order Bessel function of the first kind. The argument of this term contains the classical drift velocity through the kinetic energy

$$\alpha = \tfrac{1}{2}m\bar{v}^2. \quad (4.3)$$

This term affects the oscillations when the electric field is rather strong. Assuming for the Fermi energy the typical value of 10^{-12} erg, the electric field must satisfy the condition

$$\sqrt{(E_1^2 + E_2^2)}/H \gtrsim 10^{-7}. \quad (4.4)$$

Keep in mind that the magnetic field is about 10^4 Oe so that the argument of the term $1/\sinh(r\pi^2 kT/\mu H)$ is of the order of unity. Under condition (4.4) the electric field does not disturb the oscillations of the magnetic field very much.

We now calculate the magnetic susceptibility. We assume that $\zeta/\mu H$ is the large parameter of the free energy (4.1). Differentiation of the Bessel function contributes a term proportional to $\sqrt{\alpha/\zeta}$, which has been considered as small. A straight calculation yields

$$\chi = \frac{M}{H} = \frac{m^{3/2}}{3\pi^2\hbar^2}\pi^2(2\zeta)^{1/2}\left\{1 - \frac{3\pi kT}{\mu H}\left(\frac{\zeta}{\mu H}\right)^{1/2}\sum_{r=1}^{\infty} J_0\left(\frac{2r\pi}{\mu H}\sqrt{\zeta\alpha}\right) \times \sin\left[\frac{\zeta+\alpha}{\mu H} r\pi - \tfrac{1}{4}\pi\right]\frac{(-1)^r}{r^{1/2}\sinh(r\pi^2 kT/\mu H)}\right\}. \quad (4.5)$$

As is easily seen for zero electric field we refine the De Haas–Van Alphen effect. As far as the electric field satisfies the condition $\sqrt{(E_1^2+E_2^2}/H^2 < 10^{-7}$, the electric field does not disturb the oscillations of the magnetic susceptibility very much. The period of the oscillations is determined by the factor $\sin(\frac{1}{4}\pi - (\zeta+\alpha)r\pi/\mu H)$ while Bessel function becomes unity. That is, the effect of the electric field is limited in the shifting of the sine argument, and this happens because the energy of the electron has been increased by an amount α. For larger values of the electric field we have to take into account the degeneracy of the energy eigenvalues. As a consequence the term $\exp\{\alpha\mu H b^2 \coth(\mu H b)\}$ is added to the partition function and the calculation of the magnetic susceptibility now gives the oscillating term $J_0((2r\pi/\mu H)\sqrt{\alpha\zeta})$ which belongs to that part of the partition function. The oscillations of the magnetic susceptibility now become very complicated as in addition the kinetic energy α depends on the strength of the magnetic field.

5. Unstable states

We now study the case $\sqrt{\alpha/\zeta} \gg 1$. Although we can approach this case with stronger electric fields, we have to assume the Fermi energy less than its typical value in order not to approach relativistic velocities. The parameter α at relativistic velocities becomes as large as 10^{-8} erg.

To find out how the constant term of (3.21) behaves for large values of the parameter $\sqrt{\alpha/\zeta}$ we perform the contour integration:

$$\frac{1}{\pi \mathrm{i}} \oint \frac{r\pi^2(1+t\,\mathrm{i})kT/\mu H}{\sinh(r\pi^2(1+t\,\mathrm{i})kT/\mu H)} \frac{1}{t} \exp\left\{\frac{r\pi}{\mu H}\sqrt{\alpha\zeta}\left(\sqrt{\frac{\zeta}{\alpha}}\,t - \sqrt{\frac{\alpha}{\zeta}}\frac{1}{t}\right)\right\} \mathrm{d}t$$

by the help of the saddle-points method, notice that

$$\sqrt{\frac{\alpha}{\zeta}} \gg 1 \rightarrow \frac{r\pi\sqrt{\alpha\zeta}}{\mu H} \gg \frac{\zeta r\pi}{\mu H} \gg 1 .$$

The two saddle points are $\pm\sqrt{(\alpha/\zeta)}\,\mathrm{i}$ which are exactly the points where the decay begins. The contours of integration are shown in fig. 1 and the corresponding integrals which are of Hankel's type are as follows:

$$\frac{1}{\sqrt{\frac{r\pi^2}{\mu H}\sqrt{|\alpha\zeta|}}} \frac{r\pi^2\sqrt{\frac{\alpha}{\zeta}}\frac{kT}{\mu H}}{\sinh\left(r\pi^2\sqrt{\frac{\alpha}{\zeta}}\frac{kT}{\mu H}\right)} \sin\left(\frac{(\sqrt{\zeta}+\sqrt{\alpha})^2}{\mu H} r\pi\right), \quad \text{contour } C_1, \tag{5.1}$$

$$\frac{1}{\sqrt{\frac{r\pi^2}{\mu H}\sqrt{|\alpha\zeta|}}} \frac{r\pi^2\sqrt{\frac{\alpha}{\zeta}}\frac{kT}{\mu H}}{\sinh\left(r\pi^2\sqrt{\frac{\alpha}{\zeta}}\frac{kT}{\mu H}\right)} \cos\left(\frac{(\sqrt{\zeta}-\sqrt{\alpha})^2}{\mu H} r\pi\right), \quad \text{contour } C_2, \tag{5.2}$$

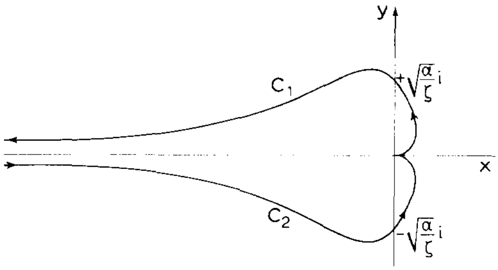

Fig. 1. Contours of integration.

where it has been assumed that the values $1-\sqrt{\alpha/\zeta}$ and $1+\sqrt{\alpha/\zeta}$ are equal to $\sqrt{\alpha/\zeta}$ in magnitude.

Summation of (5.1) and (5.2) gives the free energy

$$F - N\zeta = -2\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} F_0(\zeta) + \frac{\sqrt{2}\, m^{3/2}}{\pi^2\hbar^3} kT(\mu H)^{3/2} \frac{(-1)^r}{\sqrt{\frac{r\pi^2}{\mu H}}\sqrt{|\alpha\zeta|}} \frac{\alpha/\zeta}{r^{3/2}\sinh\left(r\pi^2\sqrt{\frac{\alpha}{\zeta}}\frac{kT}{H}\right)} \times \sin\left\{\left(\frac{(\sqrt{\zeta}+\sqrt{\alpha})^2}{\mu H} r\pi\right) + \cos\left(\frac{(\sqrt{\zeta}+\sqrt{\alpha})^2}{\mu H} r\pi\right)\right\}, \tag{5.3}$$

Comparing this formula with (4.1) we can easily check that the passage from $\sqrt{\alpha/\zeta} \ll 1$ to $\sqrt{\alpha/\zeta} \gg 1$ is done through the asymptotic formula of Bessel functions, namely

$$J_0(x) \approx \sqrt{(2/\pi x)}\cos(x - \pi/4)\,. \tag{5.4}$$

To obtain the magnetic susceptibility, we differentiate only sine and cosine terms of (5.3) taking $\alpha/\mu H$ as a large parameter. We find

$$x = \frac{m^{3/2}}{3\pi^2\hbar^3}\mu^2(2\zeta)^{1/2}\left\{1 + \frac{9\,kT}{\mu H}\left(\frac{|\alpha|}{\zeta}\right)^{5/4}\sum_{r=1}^{\infty}\frac{(-1)^r}{r\sinh r\pi^2\sqrt{\frac{|\alpha|}{\zeta}\frac{kT}{\mu H}}} \times \left[\cos\left(\frac{(\sqrt{\zeta}+\sqrt{\alpha})^2}{\mu H} r\pi\right) - \sin\left(\frac{(\sqrt{\zeta}-\sqrt{\alpha})^2}{\mu H} r\pi\right)\right]\right\}. \tag{5.5}$$

Now if the oscillating terms in (5.5) are appreciable in magnitude, it is necessary for $\pi^2\sqrt{(\alpha/\zeta)}\,(kT/\mu H)$ to be of the order unity. So the magnetic field must be very strong, $\mu H \gg kT$, but not necessarily stronger than the atomic fields $H = 10^8$–10^9 Oe. The electron rotates, because of the strong magnetic field, in an orbit close enough to the Fermi surface, so that the electric field performs the ionization in a quantum-mechanical manner. This effect does not have a quasi-classical character, because of the condition

$E \gg \alpha \gg \zeta$. As a consequence, we find for the magnetic susceptibility a difference of quasi-periodical terms which differ by $\pi/2$ and by the sign of the charge e. Finally, we note that in the two approximate cases, the resulting series are rapidly convergent with respect to r, and this justifies our earlier statements.

6. Conclusion

In this paper the effect of a sufficiently weak electric field, $\sqrt{\alpha/\epsilon} \ll 1$, on the oscillations of the magnetic susceptibility has been studied. It has been proved that when the electric field is very weak, $E/H \ll 10^{-7}$, the contribution is limited to a small shifting in the argument of the periodical terms. Over this value a new oscillating term is added, coming through the Bessel function, which is due to the interaction of the magnetic field with the crossed component of the electric field. When the fields become stronger, $\mu H \gg kT$ and $v_F \ll c\sqrt{(E_1^2 + E_2^2)}/H^2$ (v_F the Fermi velocity), the magnetic susceptibility becomes the difference of two quasi-periodical terms which have a difference of $\pi/2$ in their phases and an opposite sign of the charge e. However, the authors think that no experimental effects have yet been discovered, such that manifest these properties, in spite of the optimistic results.

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