PROPAGATOR WITH FRICTION IN QUANTUM MECHANICS

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In this paper we calculate the propagator for quantum-mechanical systems with friction. For the case where the friction is a linear function of the velocity with a friction constant γ we can calculate exact propagators of quadratic form.

Recently, Moreira [1] and Khandekar and Lawande [2,3] calculated the exact propagator for a quadratic lagrangian with the help of the Van Vleck–Pauli formula, which, in one dimension, reads

$$K(q'',t'';q',t')=\left(\frac{\mathrm{i}}{2\pi\hbar}\frac{\partial^2 S}{\partial q''\partial q'}\right)^{1/2}\exp[(\mathrm{i}/\hbar)S(q'',t'';q',t')]\ , \tag{1}$$

where S is the classical action, which corresponds to the given lagrangian with friction, from the space–time point (q',t') to (q'',t''). Instead of eq. (1) the following definition of the propagator can be used [3]:

$$K(q'',t'';q',t')=\sum_n \Psi_n^*(q',t')\Psi_n(q'',t''), \tag{2}$$

where the wave function $\Psi_n(q,t)$ is defined in ref. [3].

For the above propagator the Hamilton operator depends on the time, and in addition it is assumed that there exists a hermitian invariant operator $I(t)$ which does not involve time differentiation. This is the case when the hamiltonian is of quadratic form with parameters which are a function of time and with friction terms.

For the case

$$\Psi_n(q,t)=\exp[-(\mathrm{i}/\hbar)E_n t]\ U_n(q), \tag{3}$$

where $U_n(q)$ is the eigenfunction of the hamilton operator, eq. (2) yields the well-known definition of the propagator [4]

$$K(q'',t'',q',t')=\sum_n U_n^*(q'')U_n(q')\exp[-(\mathrm{i}/\hbar)E_n(t''-t')]\ . \tag{4}$$

In what follows the general eq. (2) will be used for the calculation of the propagator of all quadratic hamiltonians with friction and external fields. The case of the classical harmonic oscillator will be considered first.

1. Damped harmonic oscillator.

$$H=(p^2/2m)\mathrm{e}^{-\gamma t}+\tfrac{1}{2}m\omega^2q^2\mathrm{e}^{\gamma t}\ . \tag{5}$$

The solution of the Schrödinger equation is given in refs. [5–7]:

$$\Psi_n(q,t) = (m\Omega/\pi\hbar)^{1/4}(2^n n!)^{-1/2}\exp\{[\tfrac{1}{4}\gamma - i\Omega(n+\tfrac{1}{2})]\,t - (m/2\hbar)(\Omega + \tfrac{1}{2}i\gamma)e^{\gamma t}q^2\}\mathcal{H}_n[(m\Omega/\hbar)^{1/2}e^{\gamma t/2}q]\;, \quad (6)$$

where $\Omega^2 = \omega^2 - \frac{1}{2}\gamma^2 > 0$, γ is the friction constant and $\mathcal{H}_n(x)$ are Hermite polynomials.

With the aid of eq. (2) and after a certain amount of algebra the propagator of the damped harmonic oscillator can be expressed as

$$K(q'',t'';q',t') = \frac{e^{\gamma(t''+t')/4}}{[(2\pi i\hbar/m\Omega)\sin\Omega(t''-t')]^{1/2}}$$

$$\times \exp\left\{\frac{im\Omega}{2\hbar}\left[\cot\Omega(t''-t')(e^{\gamma t''}q''^2 + e^{\gamma t'}q'^2) - \frac{2q''q'e^{\gamma(t''+t')/2}}{\sin\Omega(t''-t')}\right] + \frac{i\gamma m}{4\hbar}(e^{\gamma t''}q''^2 - e^{\gamma t'}q'^2)\right\}. \quad (7)$$

The above eq. (7) for $\gamma = 0$ gives the propagator of the harmonic oscillator [4].

2. Forced and damped oscillator.

$$H = (p^2/2m)e^{-\gamma t} + (\tfrac{1}{2}m\omega^2q^2 - qF(t))e^{\gamma t}\,. \quad (8)$$

The corresponding solution of the Schrödinger equation is given by Kerner [1]:

$$\Psi_n(q,t) = \left(\frac{m\Omega}{\pi\hbar}\right)^{1/4}(2^n n!)^{-1/2}\exp\left\{-\frac{i}{\hbar}\left[\int\Delta(t)dt + E_n t + \tfrac{1}{4}m\gamma e^{\gamma t}q^2 - q\,(p_0(t) + \tfrac{1}{2}m\gamma q_0(t)e^{\gamma t})\right]\right\}$$

$$\times \exp\{-(m\Omega/2\hbar)[q - q_0(t)]^2 e^{\gamma t}\}\mathcal{H}_n\{(m\Omega/\hbar)^{1/2}[q - q_0(t)]\,e^{\gamma t/2}\}\,, \quad (9)$$

where

$$\Delta(t) = L_0 + \tfrac{1}{2}m\gamma\dot{q}_0(t)q_0(t)e^{\gamma t} + \tfrac{1}{4}i\gamma, \quad E_n = \hbar\Omega(n+\tfrac{1}{2}), \quad p_0(t) = m\dot{q}_0(t)e^{\gamma t}, \quad \Omega^2 = \omega^2 - \tfrac{1}{4}\gamma^2 > 0\,, \quad (10)$$

and L_0 represents the classical lagrangian for the damped but unforced motion as a function in time of the damped and forced position and velocity. The position coordinate $q_0(t)$ satisfies the classical equation:

$$m\ddot{q}_0 + \gamma\dot{q}_0 + m\omega^2 q_0 = F(t)\,. \quad (11)$$

With the aid of the generalized propagator, given by eq. (2), and after a certain amount of algebra we finally find

$$K(q'',t'',q',t') = \left[\frac{m\Omega}{2\pi i\hbar\sin\Omega(t''-t')}\right]^{1/2}\exp\left\{\frac{im\Omega}{2\hbar\sin\Omega(t''-t')}\,[(q''-q_0(t''))^2e^{\gamma t''}\right.$$

$$\left. + (q'-q_0(t'))^2e^{\gamma t'}\cos\Omega(t''-t') - 2(q''-q_0(t''))\,(q'-q_0(t'))\,e^{\gamma(t''+t')/2}]\right\}$$

$$\times \exp\left\{-\frac{i}{\hbar}\int[\Delta(t'')dt'' - \Delta^*(t')\,dt'] - \frac{im\gamma}{4\hbar}(q''^2e^{\gamma t''} - q'^2e^{\gamma t'})\right.$$

$$\left. -\frac{i}{\hbar}[q''(p_0(t'') + \tfrac{1}{2}m\gamma q_0(t'')e^{\gamma t''}) - q'(p_0(t') + \tfrac{1}{2}m\gamma q_0(t')e^{\gamma t'})]\right\}. \quad (12)$$

For the case where there is no external force, that is, $F(t) = 0$, eq. (11) gives $q_0 = 0$ and formula (12) coincides with formula (7).

3. Applied electric field and friction. The case of electric field and friction is the most interesting. The solution of the Schrödinger equation for an electric field has been given by Buch and Denman [8]. According to Husimi [9] the wave function $\Psi_k(q,t)$ with friction and for an applied electric field $\mathcal{E}(t)$ has the following form:

$$\Psi_k(q,t) = \exp\left\{ik\left[q - \frac{e}{m}\int^t \mathrm{e}^{-\gamma\tau}\int^\tau \mathcal{E}(V)\mathrm{e}^{\gamma V}\,\mathrm{d}V\right] + \frac{\mathrm{i}\hbar}{2m\gamma}k^2\mathrm{e}^{-\gamma t} + \mathrm{i}\frac{e}{\hbar}\left[\int^\tau \mathcal{E}(\tau)\mathrm{e}^{\gamma\tau}\,\mathrm{d}\tau\right]q\right\}$$

$$\times \exp\left\{-\frac{\mathrm{i}e^2}{2m\hbar}\int^t \mathrm{e}^{-\gamma\tau}\left[\int^\tau \mathcal{E}(V)\,\mathrm{e}^{\gamma V}\,\mathrm{d}V\right]^2 \mathrm{d}\tau\right\}. \tag{13}$$

According to eq. (2) the propagator becomes:

$$K(q'',t'';q',t') = 2\exp\left\{\frac{\mathrm{i}e}{\hbar}\left[\int^{t''} \mathcal{E}(\tau)\mathrm{e}^{\gamma\tau}\,\mathrm{d}\tau\, q'' - \int^{t'} \mathcal{E}(\tau)\mathrm{e}^{\gamma\tau}\,\mathrm{d}\tau\, q'\right] - \frac{\mathrm{i}e^2}{2m\hbar}\int_{t'}^{t''} \mathrm{e}^{-\gamma\tau}\left[\int^\tau \mathcal{E}(V)\mathrm{e}^{\gamma V}\,\mathrm{d}V\right]^2 \mathrm{d}\tau\right\}$$

$$\times \int_0^\infty \exp\left\{\frac{\mathrm{i}\hbar}{2m\gamma}(\mathrm{e}^{-\gamma t''} - \mathrm{e}^{-\gamma t'})k^2 + \mathrm{i}k\left[q'' - q' - \frac{e}{m}\left(\int_{t'}^{t''} \mathrm{e}^{-\gamma\tau}\int^\tau \mathcal{E}(V)\mathrm{e}^{\gamma V}\,\mathrm{d}V\right)\right]\right\}\mathrm{d}k; \tag{14}$$

the above integral is of Frensel type [10] and eq. (14) can finally be written as:

$$K(q'',t'';q',t') = \left[\frac{-m\gamma}{2\pi\mathrm{i}\hbar(\mathrm{e}^{-\gamma t''} - \mathrm{e}^{-\gamma t'})}\right]^{1/2} \exp\left\{-\frac{\mathrm{i}m\gamma}{2\hbar(\mathrm{e}^{-\gamma t''} - \mathrm{e}^{-\gamma t'})}\left[q'' - q' - \frac{e}{m}\left(\int_{t'}^{t''}\mathrm{e}^{-\gamma\tau}\int^\tau \mathcal{E}(V)\mathrm{e}^{\gamma V}\mathrm{d}V\right)\right]^2\right\}$$

$$\times \exp\left\{\frac{\mathrm{i}e}{\hbar}\left[\int^{t''} \mathcal{E}(\tau)\mathrm{e}^{\gamma\tau}\,\mathrm{d}\tau\, q'' - \int^{t'} \mathcal{E}(\tau)\mathrm{e}^{\gamma\tau}\mathrm{d}\tau\, q'\right] - \frac{\mathrm{i}e^2}{2m\hbar}\int_{t'}^{t''} \mathrm{e}^{-\gamma\tau}\left[\int^\tau \mathcal{E}(V)\mathrm{e}^{\gamma V}\mathrm{d}V\right]^2 \mathrm{d}\tau\right\}. \tag{15}$$

The case where the function $\mathcal{E}(t)$ is constant, $\mathcal{E}(t) = \mathcal{E}_0$, has been studied by Moreira [1].

4. The forced harmonic oscillator.

$$H = -(\hbar^2/2m)\,\partial^2/\partial q^2 + \tfrac{1}{2}m\,\omega(t)^2 q^2 - f(t)q\,. \tag{16}$$

According to Husimi [9] the Schrödinger equation with a hamiltonian given by eq. (16) has a solution of gaussian type, namely

$$\Psi_k(q,t) = \exp\{(\mathrm{i}/2\hbar)[a(t)q^2 + 2\tilde{b}(t)q + \tilde{c}(t)] - (\mathrm{i}k^2/2m\hbar)A(t) + (\mathrm{i}k/\hbar)\,[qB(t) + \Gamma(t)]\}\;, \tag{17}$$

where the function $a(t)$ satisfies the following Ricatti equation:

$$m^{-1}\,\mathrm{d}a/\mathrm{d}t = -a^2/m^2 - \omega^2(t)\,, \tag{18}$$

$$\tilde{b}(t) = \exp\left[-\frac{1}{m}\int^t a(\tau)\mathrm{d}\tau\right]\int^t f(\tau)\exp\left[\frac{1}{m}\int^\tau a(\tau')\,\mathrm{d}\tau'\right]\mathrm{d}\tau\,, \tag{19}$$

$$\tilde{c}(t) = \frac{\mathrm{i}\hbar}{m}\int^t a(\tau)\mathrm{d}\tau - \frac{1}{m}\int^t \mathrm{d}\tau\,\exp\left[-\frac{2}{m}\int^\tau a(\tau')\mathrm{d}\tau'\right]\left\{\int^\tau f(\tau')\exp\left[\frac{1}{m}\int^\tau a(\tau'')\mathrm{d}\tau''\right]\mathrm{d}\tau'\right\}^2, \tag{20}$$

$$A(t) = \int^t \exp\left[-\frac{2}{m}\int^\tau t\,a(\tau')\mathrm{d}\tau'\right]\mathrm{d}\tau\,,\quad B(t) = \exp\left[-\frac{1}{m}\int^t a(\tau)\mathrm{d}\tau\right], \tag{21,22}$$

$$\Gamma(t) = \frac{1}{m}\int^t \mathrm{d}\tau\exp\left[-\frac{2}{m}\int^\tau a(\tau')\mathrm{d}\tau'\right]\left\{\int^\tau f(\tau')\exp\left[\frac{1}{m}\int^{\tau'} a(\tau'')\,\mathrm{d}\tau''\right]\mathrm{d}\tau'\right\}\;. \tag{23}$$

By means of eq. (2) the propagator can be expressed as:

$$K(q'',t'';q',t') = \exp\{(\mathrm{i}/2\hbar)[(a(t'')q''^2 + 2\tilde{b}(t'')q'' + \tilde{c}(t'')) - (a(t')q'^2 + 2\tilde{b}(t')q' + \tilde{c}(t'))]\}$$

$$\times \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \exp\{-(\mathrm{i}k^2/2m\hbar)[A(t'') - A(t')] + (\mathrm{i}k/\hbar)\,[q''B(t'') - q'B(t') + \Gamma(t'') - \Gamma(t')]\}\,\mathrm{d}k\ . \quad (24)$$

If we set $k \to hk$ in the integral of the r.h.s. of eq. (24), this is again of Frensel type and the propagator is now given by

$$K(q'',t'',q',t')\exp\{(\mathrm{i}/2\hbar)[(a(t'')q''^2 + 2\tilde{b}(t'')q + \tilde{c}(t')) - (a(t')q'^2 + 2\tilde{b}(t')q' + \tilde{c}(t'))]$$

$$\times \frac{m^{1/2}}{\{2\pi\mathrm{i}\hbar[A(t'') - A(t')]\}^{1/2}} \exp\left\{-\frac{\mathrm{i}m}{2\hbar^2}\frac{[q''B(t'') - q'B(t') + \Gamma(t'') - \Gamma(t')]^2}{A(t'') - A(t')}\right\}\ . \quad (25)$$

In the same way, we can study the case of the damped and forced harmonic oscillator. The hamiltonian has now the form:

$$H = -(\hbar^2/2m)\mathrm{e}^{-\gamma t}\partial^2/\partial q^2 + \mathrm{e}^{\gamma t}\,[\tfrac{1}{2}m\,\omega^2(t)q^2 - f(t)q]\ . \quad (26)$$

With the help of the following contact transformation [7]:

$$q \to \mathrm{e}^{-\gamma t/2}Q, \quad \Psi(q,t) = \exp[\tfrac{1}{4}\gamma t + (\mathrm{i}m\gamma/4\hbar)\,\mathrm{e}^{\gamma t}q^2]\,\Psi_k(Q,t)\ , \quad (27,28)$$

the wave function satisfies the Schrödinger equation

$$\mathrm{i}\hbar\,\partial\Psi/\partial t = -(\hbar^2/2m)\,\partial^2\Psi/\partial Q'^2 + [\tfrac{1}{2}m\,\Omega(t)^2Q'^2 - F(t)Q']\,\Psi\ , \quad (29)$$

where

$$\Omega(t)^2 = \omega(t)^2 - \tfrac{1}{4}\gamma^2, \quad F(t) = \mathrm{e}^{\gamma t/2}f(t)\ . \quad (30)$$

So we can easily find the propagator:

$$\tilde{K}(q'',t'',q',t') = \exp[\tfrac{1}{4}\gamma(t'' + t') + (\mathrm{i}m\gamma/4\hbar)\,(\mathrm{e}^{\gamma t''}q''^2 - \mathrm{e}^{\gamma t'}\,q'^2)]\,K(Q'',t'';Q',t')\ , \quad (31)$$

where $K(Q'',t'';Q',t') = K(\mathrm{e}^{\gamma t''/2}q'',t'',\mathrm{e}^{\gamma t'/2}q',t')$ is the propagator (25).

The evaluation of the propagator for the hamiltonian (26) has been carried out by Khandekar and Lawande [2, 3] by means of the known method of path integrals [11], which has been used for the same purpose by other authors [12,13].

5. Damped harmonic oscillator in a uniform magnetic field. Another interesting case is that of the damped harmonic oscillator in a uniform magnetic field, which we will study now.

Accoding to Jannussis et al. [14], the Hamilton operator is given by the relation

$$H(t) = (1/2m)\,[\boldsymbol{p} + (e/c)\boldsymbol{H}(t)\times\boldsymbol{q}]^2\,\mathrm{e}^{-\gamma t} + \tfrac{1}{2}\mathrm{e}^{\gamma t}m\omega^2(q_1^2 + q_2^2 + q_3^2)\ , \quad (32)$$

where $\boldsymbol{H}(t) = \boldsymbol{H}\mathrm{e}^{\gamma t}$ and $\boldsymbol{H}$ is the constant intensity of the magnetic field. Using the contact transformation $\boldsymbol{q} = \mathrm{e}^{-\gamma t/2}\boldsymbol{Q}$, the solution of the time-dependent Schrödinger equation has the following form:

$$\Psi(\boldsymbol{Q},t) = \exp[\tfrac{3}{4}\gamma t - (\mathrm{i}m\gamma/4\hbar)Q^2]\,F_1(Q_1,Q_2,t)F_3(Q_3,t),$$

where

$$F_1(Q_1,Q_2,t) = \left(\frac{m\Omega}{\pi\hbar}\right)^{1/2}\frac{(n!)^{1/2}}{[(n+l)!]^{1/2}}\exp\left(-\frac{m\Omega}{2\hbar}r^2\right)\left(\frac{m\Omega}{\hbar}r^2\right)^{l/2}L_n^l\,[(m\Omega/\hbar)r^2]\,\mathrm{e}^{\pm\mathrm{i}l\vartheta}\mathrm{e}^{-(\mathrm{i}/\hbar)tE_{nl}}, \quad (33)$$

$$F_3(Q_3,t)=\left(\frac{m\Omega}{\pi\hbar}\right)^{1/2}(2^m m!)^{-1/2}\exp\left(-\frac{m\Omega_3}{2\hbar}Q_3^2\right)\mathcal{H}_m[(m\Omega_3/\hbar)Q_3]\,\mathrm{e}^{-(\mathrm{i}/\hbar)tE_m}\,,$$

and $L_n^l(x)$ are the generalized Laguerre polynomials. The above solutions have been expressed in polar coordinates, that is $Q_1=r\cos\vartheta, Q_2=r\sin\vartheta, Q_3=Q_3$. The energy eigenvalues are as follows:

$$E_n=\hbar\Omega(2n+l+1)\pm\hbar\omega_\mathrm{L}l,\quad \Omega^2=\omega_\mathrm{L}^2+\omega^2-\tfrac{1}{4}\gamma^2,\quad \omega_\mathrm{L}=eH/2mc\,, \tag{34}$$

$$E_m=\hbar\Omega_3(m+\tfrac{1}{2}),\quad \Omega_3^2=\omega^2-\tfrac{1}{4}\gamma^2\,.$$

The calculation of the propagator proceeds now easily through relation (2). After a certain amount of algebra we obtain

$$K(\boldsymbol{q}'',t'';\boldsymbol{q}',t')=\left[\frac{m\Omega_3}{2\mathrm{i}\pi\hbar\sin\Omega_3(t''-t')}\right]^{1/2}\frac{m\Omega\mathrm{e}^{3\gamma(t''+t')/4}}{2\mathrm{i}\pi\hbar\sin\Omega(t''-t')}\exp\left\{-\frac{\mathrm{i}m\gamma}{4\hbar}(\mathrm{e}^{\gamma t'}q'^2-\mathrm{e}^{\gamma t''}q''^2)\right.$$
$$+\frac{\mathrm{i}m\Omega_3}{2\hbar\sin\Omega_3(t''-t')}[\cos\Omega_3(t''-t')(\mathrm{e}^{\gamma t''}q_3''^2+\mathrm{e}^{\gamma t'}q_3'^2)-2\mathrm{e}^{\gamma(t''+t')/2}q_3''^2q_3'^2]$$
$$+\frac{\mathrm{i}m\Omega}{2\hbar\sin\Omega(t''-t')}\{\cos\Omega(t''-t')\,[\mathrm{e}^{\gamma t''}(q_1''^2+q_2''^2)+\mathrm{e}^{\gamma t'}(q_1'^2+q_2'^2)] \tag{35}$$
$$\left.-2\cos\omega_\mathrm{L}(t''-t')\mathrm{e}^{\gamma(t''+t')/2}(q_1''q_1'+q_2''q_2')-2\sin\omega_\mathrm{L}(t''-t')\,\mathrm{e}^{\gamma(t''+t')/2}(q_1'q_2''-q_2'q_1'')\}\right\}.$$

It is clear from the result of the present work that the generalized propagator defined by eq. (2) considerably simplifies the evaluation of the propagator for quadratic forms with friction; moreover, it can be used for time-dependent hamiltonians of a more general form.

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