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Quantum damped harmonic oscillator on non - commuting plane

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Abstract

We study the deformed harmonic oscillator in the presence of friction. We use the following time dependent Hamiltonian:

$$\mathcal{H}(\hat{p}_1, \hat{q}_1, t) = e^{-2\gamma t}\,\frac{1}{2m}\,\hat{p}_1^2 + e^{2\gamma t}\,\frac{m\omega^2}{2}\,\hat{q}_1^2$$

Dissipation arises from interactions between the observed system and another one, often called the reservoir or the bath, into which the energy flows in an irreversible manner. A method, used at first by H. Bateman to apply the usual canonical quantization method, is based upon the procedure of doubling the number of degrees of freedom so as to deal with an effective isolated system.

In this paper we use the following Hamiltonian ($m = 1$):

$$\mathcal{H}(\hat{p}_1, \hat{q}_1, t) - \mathcal{H}(\hat{p}_2, \hat{q}_2, -t) = \frac{1}{2}e^{2\gamma t}\left(\hat{p}_1^2 - \omega_2^2\hat{q}_2^2\right) - \frac{1}{2}\,e^{-2\gamma t}\left(\hat{p}_2^2 - \omega_1^2\hat{q}_1^2\right)$$

which is actually a damped harmonic oscillator coupled to its time - reserved image. The two Hamiltonians do not commute and the basic operators satisfy the following commutation relations of non commuting geometry.

$$[\hat{p}_1, \hat{p}_2] = i\lambda \qquad [\hat{q}_1, \hat{q}_2] = i\theta \qquad [\hat{q}_1, \hat{p}_1] = i\hbar \qquad [\hat{q}_2, \hat{q}_2] = i\hbar$$

where λ, θ are real parameters.

We calculate the time evolution operator and we find the exact propagator of the system.

The resulting propagator depends on the deformed parameter μ and is a two - dimensional Gaussian type distribution function of the commuting observables $\tau_1 = q_1$ and $\tau_2 = q_2 - (\theta/\hbar)p_1$. The oscillating terms depend on the frequencies Ω_1 and Ω_2.

$$\hat{\omega}_{1,2} = \frac{1}{2}\sqrt{(\omega_1+\omega_2)^2 - (\lambda+\omega_1\omega_2\theta)^2/\hbar^2} \pm \frac{1}{2}\sqrt{(\omega_1-\omega_2)^2 - (\lambda-\omega_1\omega_2\theta)^2/\hbar^2}$$

$$\Omega_1 = \sqrt{\hat{\omega}_1^2 - \gamma^2}, \quad \Omega_2 = \sqrt{\hat{\omega}_2^2 - \gamma^2} \qquad \mu = 1 - \frac{\lambda\theta}{\hbar^2} = \frac{\hat{\omega}_1}{\omega_1}\frac{\hat{\omega}_2}{\omega_2}$$

We investigate the thermodynamic properties of the system using the standard canonical density matrix. We find the statistical distribution function and the partition function.

We calculate the specific heat for the limiting case of critical damping, where the frequencies of the system vanish $\Omega_1 = 0$, $\Omega_2 = 0$, which can be achieved if $\omega_1 = \pm\omega_2 = \gamma/\sqrt{\mu} = \sqrt{\lambda/\theta}$. The specific heat c of this system posseses some singularities (Fig.1) which disappear in the classical limit $\hbar \to 0$ (Fig.2). The values of T where the denominator of the partition function becomes zero, are

$$T_n \cong \frac{\hbar\gamma}{2k}\frac{1}{1.16556 + n\pi}, \quad n = 0, 1, 2, \cdots \qquad T_n = \frac{\hbar\gamma}{2k}\frac{1}{n\pi}, \quad n = 1, 2, \cdots$$

where k is the Boltzmann constant and $T_0 = \frac{\hbar}{k}\gamma = \frac{\hbar}{k}\sqrt{\frac{\lambda\mu}{\theta}}$.

Finaly we study the case where the the deformed parameter μ becomes zero. The propagator is again a Gaussian type distribution but now it is a function of the three commuting observables $\tau_1 = q_1$, $\tau_2 = q_2 - (\theta/\hbar)p_1$ and $\pi_2 = p_2 + (\lambda/\hbar)q_1$. The propagator depends on the following parameters and frequencies

$$\hbar = \sqrt{\lambda\theta}, \quad \sigma = \sqrt{\frac{\lambda}{\theta}}, \quad \hat{\omega} = \frac{1}{\sigma}\sqrt{(\sigma^2 - \omega_2^2)(\omega_1^2 - \sigma^2)}, \quad \Omega = \sqrt{\hat{\omega}^2 - \gamma^2}$$

The specific heat has some singularities and also some zeros for the various values of the parameters (Fig.3) and (Fig.4).

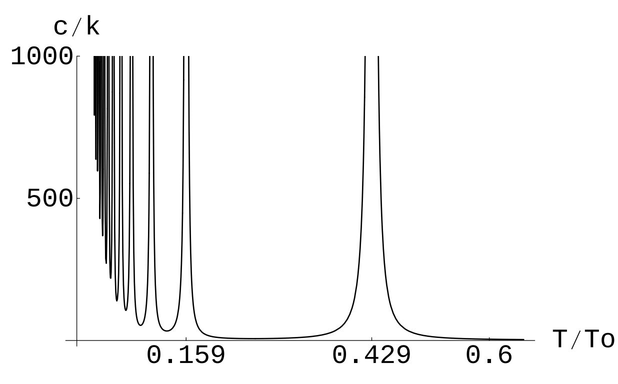

Figure 1: Critical damping. The specific heat for $T_0 = 1$.

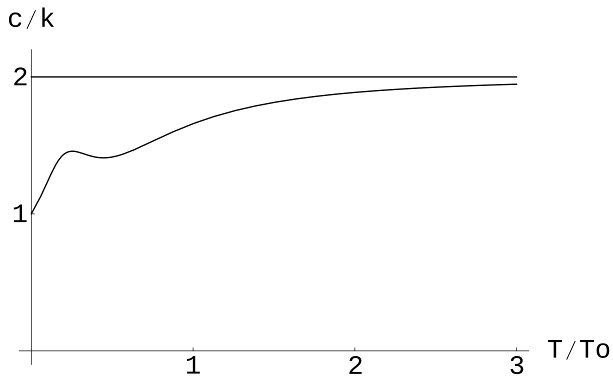

Figure 2: Critical damping. The specific heat for $T_0 = i$.

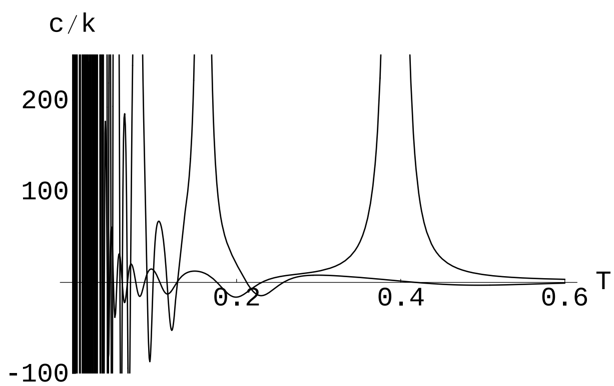

Figure 3: The case $\mu = 0$. The specific heat for $\gamma = 1$ and $\Omega = 2\,i$, $\Omega = 0.8\,i$.

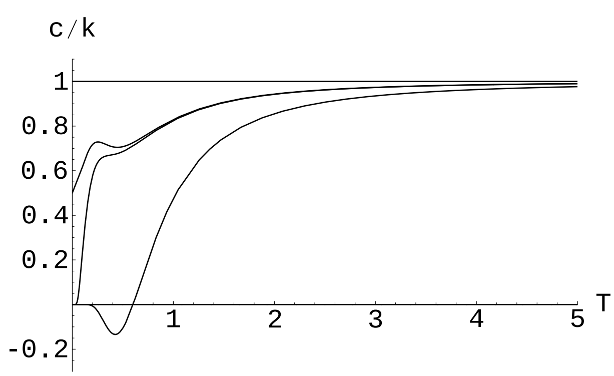

Figure 4: The case $\mu = 0$. The specific heat for $\gamma = i$ and $\Omega = 0$, $\Omega = 0.2$, $\Omega = 2$.