DENSE MORPHISMS OF MONADS

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Abstract. Given an arbitrary locally finitely presentable category $\mathcal{K}$ and finitary monads $\mathbb{T}$ and $\mathbb{S}$ on $\mathcal{K}$, we characterize monad morphisms $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ with the property that the induced functor $\alpha_* : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}^{\mathbb{S}}$ between the categories of Eilenberg-Moore algebras is fully faithful. We call such monad morphisms dense and give a characterization of them in the spirit of Beth's definability theorem: α is a dense monad morphism if and only if every $\mathbb{T}$-operation is explicitly defined using $\mathbb{S}$-operations. We also give a characterization in terms of epimorphic property of α and clarify the connection between various notions of epimorphisms between monads.

1. Introduction

We study embedding functors $\Phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are finitary varieties, such that Φ does not change the underlying sets of respective algebras. More precisely: we study situations

$$\begin{tikzcd} \mathcal{V}_1 \arrow[rr, "\Phi"] \arrow[dr, "U_1"'] & & \mathcal{V}_2 \arrow[dl, "U_2"] \\ & \mathsf{Set} & \end{tikzcd} \tag{1.1}$$

where U_1 and U_2 are underlying functors with the property that

$$\text{every } \mathcal{V}_2\text{-homomorphism between } \mathcal{V}_1\text{-algebras is a } \mathcal{V}_1\text{-homomorphism.} \tag{1.2}$$

Examples of situations (1.1) satisfying (1.2) abound — let us point out two trivial examples:

Examples 1.1.

(1) $\mathcal{V}_1$ is the variety of Abelian groups, $\mathcal{V}_2$ is the variety of all groups. That (1.2) holds is trivial: $\mathcal{V}_1$ arises as adding just the commutativity law to the equational presentation of $\mathcal{V}_2$ and such process does not affect the notion of a homomorphism.

(2) $\mathcal{V}_1$ is the variety of groups, $\mathcal{V}_2$ is the variety of monoids. Condition (1.2) holds since the inverse operation can be defined explicitly in the language of monoids. More precisely, the sentence

$$\forall x \forall y \left(y = x^{-1} \Leftrightarrow (x * y = e \wedge y * x = e) \right)$$

holds in every group $(G, *, e, (-)^{-1})$. Thus, the predicate $y = x^{-1}$ (i.e., "to be an inverse") is preserved by any monoid homomorphism.

In fact, the above example of groups and monoids is a good illustration of a general characterization of condition (1.2):

For every $\mathcal{V}_1$-operation τ, the predicate "to be τ" must be explicitly definable by a system of equations in the language of $\mathcal{V}_2$-operations.

This result is a special case covered by the famous *Beth's Definability Theorem* of model theory, see [Be].

We prove Beth's Definability Theorem in Theorem 4.3 below in a more general setting than (1.1). To be more specific, we replace the base category Set of sets and mappings by an essentially algebraic category $\mathcal{K}$ (see Definition 2.4 below) and we replace finitary varieties $\mathcal{V}_1$, $\mathcal{V}_2$ by categories $\mathcal{K}^{\mathbb{T}}$, $\mathcal{K}^{\mathbb{S}}$ of Eilenberg-Moore algebras for finitary monads $\mathbb{T}$ and $\mathbb{S}$, respectively, on the category $\mathcal{K}$, studying thus situations

$$\begin{tikzcd} \mathcal{K}^{\mathbb{T}} \arrow[rr, "\Phi"] \arrow[dr, "U^{\mathbb{T}}"'] & & \mathcal{K}^{\mathbb{S}} \arrow[dl, "U^{\mathbb{S}}"] \\ & \mathcal{K} & \end{tikzcd} \tag{1.3}$$

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where $U^{\mathbb{T}}$ and $U^{\mathbb{S}}$ denote the underlying functors. By putting $\mathcal{K} = \mathsf{Set}$, the situation (1.1) is recovered, since finitary varieties are precisely the categories of Eilenberg-Moore algebras for suitable finitary monads on Set, as proved by Fred Linton in [Li$_1$].

This level of generality has also the advantage that the situation (1.3) is equivalent to having a monad morphism

$$\alpha : \mathbb{S} \longrightarrow \mathbb{T} \tag{1.4}$$

and we may ask which property of monad morphisms singles out the property

$$\text{every } \mathbb{S}\text{-homomorphism between } \mathbb{T}\text{-algebras is a } \mathbb{T}\text{-homomorphism.} \tag{1.5}$$

We call such monad morphisms *dense*. We introduce, using the formalism of monads, a notion of explicit definability of operations and show in Theorem 4.3 below that α is dense exactly when every n-tuple of m-ary $\mathbb{T}$-operations is explicitly $\mathbb{S}$-definable. Furthermore, we characterize dense monad morphisms in terms of an orthogonality condition (Theorem 5.4 below), locating them strictly in between strong epimorphisms and epimorphisms in the category of finitary monads and their morphisms.

Organization of the Paper. In Section 2 we gather notions that we will need in the sequel. Various useful sufficient conditions for density of a monad morphism are given in Section 3. Section 4 is devoted to the theorem of Beth type characterizing dense monad morphisms, whereas in Section 5 we characterize dense monad morphisms in the category of finitary monads. Finally, in Section 6 we briefly indicate how one can state and prove the results of the paper in a yet more general setting than that of locally finitely presentable categories and finitary monads.

Related Work. Quite a few of sufficient conditions for density of morphisms of finitary monads on sets can be found in textbooks by Ernest Manes [M] (see, e.g., Exercise 6, Section 3, Chapter 3) and by Gavin Wraith [W] (Chapter 12). Beth's Definability Theorem for (possibly infinitary) varieties on sets was proved by John Isbell in [I$_2$] and our proof of Theorem 4.3 was much inspired by Isbell's approach.

2. Preliminaries

In this section we fix the (mostly standard) notation and terminology we will need later. We do not give any proofs, we refer the reader to corresponding publications instead.

2.A. **Monads, Their Morphisms and Their Algebras.** The relevance of monads to universal algebra is treated in great detail in the book [M] by Ernest Manes. Therefore we just recall the definitions, the proofs of statements below can all be found in Manes' book.

A *monad* on a category $\mathcal{K}$ is a triple $\mathbb{S} = (S, \eta^S, \mu^S)$ consisting of a functor $S : \mathcal{K} \longrightarrow \mathcal{K}$ and natural transformations $\eta^S : \mathrm{Id} \longrightarrow S$, $\mu^S : SS \longrightarrow S$ such that the following diagrams

$$\begin{tikzcd} S \arrow[r, "\eta^S S"] \arrow[dr, equal] & SS \arrow[d, "\mu^S"] & S \arrow[l, "S\eta^S"'] \arrow[dl, equal] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} SSS \arrow[r, "S\mu^S"] \arrow[d, "\mu^S S"'] & SS \arrow[d, "\mu^S"] \\ SS \arrow[r, "\mu^S"'] & S \end{tikzcd} \tag{2.1}$$

commute.

An *Eilenberg-Moore algebra* for a monad $\mathbb{S}$ on $\mathcal{K}$ (or $\mathbb{S}$-*algebra*) is a pair (A, a), where $a : SA \longrightarrow A$ is a morphism in $\mathcal{K}$ subject to commutativity of the diagrams

$$\begin{tikzcd} A \arrow[r, "\eta^S_A"] \arrow[dr, equal] & SA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} SSA \arrow[r, "Sa"] \arrow[d, "\mu^S_A"'] & SA \arrow[d, "a"] \\ SA \arrow[r, "a"'] & A \end{tikzcd} \tag{2.2}$$

An $\mathbb{S}$-*homomorphism* from (A, a) to (B, b) is a morphism $h : A \longrightarrow B$ making the square

$$\begin{tikzcd} SA \arrow[r, "Sh"] \arrow[d, "a"'] & SB \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd} \tag{2.3}$$

commutative.

Algebras for $\mathbb{S}$ and their homomorphisms form an *Eilenberg-Moore category* $\mathcal{K}^{\mathbb{S}}$ equipped with a natural *underlying functor* $U^{\mathbb{S}} : \mathcal{K}^{\mathbb{S}} \longrightarrow \mathcal{K}$ sending (A, a) to A. The functor $U^{\mathbb{S}}$ has always a left adjoint sending A to (SA, μ^S_A) — a *free* $\mathbb{S}$*-algebra* on A.

The full subcategory of $\mathcal{K}^{\mathbb{S}}$ spanned by free $\mathbb{S}$-algebras is called the *Kleisli category* $\mathcal{K}_{\mathbb{S}}$ of $\mathbb{S}$. We will denote the full embedding by

$$K_{\mathbb{S}} : \mathcal{K}_{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{S}} \tag{2.4}$$

A *monad morphism* between monads $\mathbb{S} = (S, \eta^S, \mu^S)$ and $\mathbb{T} = (T, \eta^T, \mu^T)$ on $\mathcal{K}$ is a natural transformation $\alpha : S \longrightarrow T$ making the following diagrams

$$\begin{tikzcd} S \arrow[rr, "\alpha"] & & T \\ & \mathrm{Id} \arrow[ul, "\eta^S"] \arrow[ur, "\eta^T"'] & \end{tikzcd} \qquad \begin{tikzcd} SS \arrow[r, "\alpha\alpha"] \arrow[d, "\mu^S"'] & TT \arrow[d, "\mu^T"] \\ S \arrow[r, "\alpha"'] & T \end{tikzcd} \tag{2.5}$$

commutative (where $\alpha\alpha$ denotes the horizontal composition of α with itself, i.e., $\alpha\alpha = \alpha T \cdot S\alpha = T\alpha \cdot \alpha S$).

It can be proved that monad morphisms $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ are in one-to-one correspondence with functors $\alpha_* : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}^{S}$ that commute with the underlying functors, i.e., they correspond to commutative triangles

$$\begin{tikzcd} \mathcal{K}^{\mathbb{T}} \arrow[rr, "\alpha_*"] \arrow[dr, "U^{\mathbb{T}}"'] & & \mathcal{K}^{\mathbb{S}} \arrow[dl, "U^{\mathbb{S}}"] \\ & \mathcal{K} & \end{tikzcd} \tag{2.6}$$

The functor α_* is given by $(A, a) \mapsto (A, a \cdot \alpha_A)$ on objects and will be referred to as *restriction along* α.

Unlike $U^{\mathbb{S}}$ and $U^{\mathbb{T}}$, the functor α_* need not have a left adjoint in general. In fact α_* has a left adjoint α^* if and only if coequalizers of the pairs

$$(TSA, \mu^T_{SA}) \underset{\mu^T_A \cdot T\alpha_A}{\overset{Ta}{\rightrightarrows}} (TA, \mu^T_A) \tag{2.7}$$

exist in $\mathcal{K}^{\mathbb{T}}$ for every $\mathbb{S}$-algebra (A, a). The value of $\alpha^* : \mathcal{K}^{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{T}}$ at an $\mathbb{S}$-algebra (A, a) is then the value of the above coequalizer and α^* is defined of morphisms using the universal property of coequalizers.

Example 2.1. An illustrative and very instructive example of the above concepts is the situation

$$\mathbb{S} = \text{monad of semigroups} \qquad \mathbb{T} = \text{monad of monoids}$$

More precisely, the monad $\mathbb{S} = (S, \eta^S, \mu^S)$ is given by the following data:

(1) For every set X, denote by SX the *free semigroup on* X. The assignment $X \mapsto SX$ extends naturally to mappings: for $f : X \longrightarrow Y$, the mapping $Sf : SX \longrightarrow SY$ is the free extension of f to non-empty words on X. Clearly, the identities $S\mathrm{id}_X = \mathrm{id}_{SX}$ and $S(g \cdot f) = Sg \cdot Sf$ hold for every set X and every composable pair f, g. The above can be summed up as follows: *S is an endofunctor of the category of sets and mappings.*

(2) For every set X, denote by $\eta^S_X : X \longrightarrow SX$ the embedding of generators. The collection of all η^S_X's "behaves well" w.r.t. to the renaming of generators, i.e., the square

$$\begin{tikzcd} X \arrow[r, "\eta^S_X"] \arrow[d, "f"'] & SX \arrow[d, "Sf"] \\ Y \arrow[r, "\eta^S_Y"'] & SY \end{tikzcd}$$

commutes for every map $f : X \longrightarrow Y$. Commutativity of all such squares is summed up as follows: *η^S is a natural transformation from* Id *to S.*

(3) For every set X, denote by $\mu^S_X : SSX \longrightarrow SX$ the map that "flattens" every non-empty word of non-empty words on X to a word on X by concatenation. Again, the collection of all μ^S_X's "behaves

well" w.r.t. to the renaming of generators, i.e., the square

$$\begin{tikzcd} SSX \arrow[r, "\mu^S_X"] \arrow[d, "SSf"'] & SX \arrow[d, "Sf"] \\ SSY \arrow[r, "\mu^S_Y"'] & SY \end{tikzcd}$$

commutes for every map $f : X \longrightarrow Y$. Commutativity of all such squares is summed up as follows: *μ^S is a natural transformation from SS to S.*

(4) Diagrams (2.1) commute. This can be easily verified when evaluating at every set X, i.e., the diagrams

$$\begin{tikzcd} SX \arrow[r, "\eta^S_{SX}"] \arrow[dr, equal] & SSX \arrow[d, "\mu^S_X"] & SX \arrow[l, "S\eta^S_X"'] \arrow[dl, equal] \\ & SX & \end{tikzcd} \qquad \begin{tikzcd} SSSX \arrow[r, "S\mu^S_X"] \arrow[d, "\mu^S_{SX}"'] & SSX \arrow[d, "\mu^S_X"] \\ SSX \arrow[r, "\mu^S_X"'] & SX \end{tikzcd}$$

commute, for every set X.

Thus, $\mathbb{S} = (S, \eta^S, \mu^S)$ is a monad on the category Set of sets and mappings. The monad $\mathbb{T} = (T, \eta^T, \mu^T)$ of monoids is defined in a similar way.

The axioms (2.2) then state that an $\mathbb{S}$-algebra (A, a) is precisely a set A equipped with an *action* $a : SA \longrightarrow A$ that evaluates non-empty words on A as elements of A, i.e., a turns A into a semigroup. The commutativity of the square (2.3) then expresses precisely the fact that h is a semigroup homomorphism. Thus, the Eilenberg-Moore category $\mathsf{Set}^{\mathbb{S}}$ is precisely the category of semigoups and their homomorphisms. Similarly, the category $\mathsf{Set}^{\mathbb{T}}$ is precisely the category of monoids and their homomorphisms.

The underlying functor $U^{\mathbb{S}} : \mathsf{Set}^{\mathbb{S}} \longrightarrow \mathsf{Set}$ assigns the underlying set A to every semigroup (A, a) and its left adjoint $F^{\mathbb{S}} : \mathsf{Set} \longrightarrow \mathsf{Set}^{\mathbb{S}}$ sends every set A to a free semigroup (SA, μ^S_A) on A. Thus, the Kleisli category $\mathsf{Set}_{\mathbb{S}}$ of $\mathbb{S}$ is precisely the category of free semigroups. Similar assertions can be made for $\mathbb{T}$.

If we denote, for every set X, by $\alpha_X : SX \longrightarrow TX$ the inclusion of a free semigroup on X into a free monoid on X, then it is easy to see that the collection of all α_X's is "well-behaved" w.r.t. to renaming, in other words, α is a natural transformation from S to T. Again, the axioms (2.5) are easily verified when evaluating at a set X, i.e., the diagrams

$$\begin{tikzcd} SX \arrow[rr, "\alpha_X"] & & TX \\ & X \arrow[ul, "\eta^S_X"] \arrow[ur, "\eta^T_X"'] & \end{tikzcd} \qquad \begin{tikzcd} SSX \arrow[r, "T\alpha_X \cdot \alpha_{SX}"] \arrow[d, "\mu^S_X"'] & TTX \arrow[d, "\mu^T_X"] \\ SX \arrow[r, "\alpha_X"'] & TX \end{tikzcd}$$

commute for every set X. Thus, α is a monad morphism from $\mathbb{S}$ to $\mathbb{T}$.

The triangle (2.6) expresses the fact that every monoid (A, a) can be considered as a semigroup $(A, a \cdot \alpha_A)$ when we "forget the unit".

2.B. **Equations w.r.t. a Monad.** Example 1.1(2) suggests that we will have to deal with more complex formulae than just identities between terms as it is done in classical universal algebra.

Thus, we are going to consider $\mathbb{S}$-algebras as first order structures for the first order language having $\mathbb{S}$-*equations as atomic formulae*. An $\mathbb{S}$-equation is a pair

$$\begin{tikzcd} X \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & SY \end{tikzcd}$$

of parallel morphisms. Intuitively, λ picks up an "X-tuple" of $\mathbb{S}$-terms on Y that form the left-hand sides of the respective system of equations. Similarly, ρ picks up the right-hand sides of the respective system of equations.

Suppose $x : Y \longrightarrow A$ is given, where (A, a) is an $\mathbb{S}$-algebra. We say that (A, a) *satisfies* $\lambda(x) = \rho(x)$, denoted by

$$(A, a) \models \lambda(x) = \rho(x)$$

if $x^\sharp \cdot \lambda = x^\sharp \cdot \rho$ holds where $x^\sharp : (SY, \mu^S_Y) \longrightarrow (B, b)$ denotes the unique extension of x to an $\mathbb{S}$-homomorphism (recall that (SY, μ^S_Y) is a free $\mathbb{S}$-algebra on Y).

The satisfaction of a general sentence in an $\mathbb{S}$-algebra (B, b) is defined inductively in the usual way. Examples:

(1) $(B, b) \models \forall x\, (\lambda(x) = \rho(x))$ means that $x^\sharp \cdot \lambda = x^\sharp \cdot \rho$ holds for *every* $x : Y \longrightarrow B$.
(2) $(B, b) \models \exists x\, (\lambda(x) = \rho(x))$ means that $x^\sharp \cdot \lambda = x^\sharp \cdot \rho$ holds for *some* $x : Y \longrightarrow B$.
(3) $(B, b) \models \exists x\, (\lambda(x) = \rho(x) \Rightarrow (\sigma(x) = \tau(x)))$ means that there exists some $x : Y \longrightarrow B$ such that $x^\sharp \cdot \lambda = x^\sharp \cdot \rho$ implies $x^\sharp \cdot \sigma = x^\sharp \cdot \tau$.

Example 2.2. Recall from Example 2.1 the monad $\mathbb{S}$ of semigroups. We show how to express the commutative law for semigroups as an $\mathbb{S}$-equation.

Denote by 2 the two-element set $\{x_1, x_2\}$ and let 1 denote the one-element set $\{*\}$. The mappings

$$\lambda : * \mapsto x_1 x_2 \qquad \rho : * \mapsto x_2 x_1$$

are then a parallel pair

$$1 \underset{\rho}{\overset{\lambda}{\rightrightarrows}} S2$$

thus, we defined an $\mathbb{S}$-equation.

Let (B, b) be any semigroup. Then

$$(B, b) \models \forall x\, (\lambda(x) = \rho(x))$$

holds, if, for every map $x : 2 \longrightarrow B$ (i.e., for every interpretation of x_1, x_2 in B), the equality $x^\sharp \cdot \lambda = x^\sharp \cdot \rho$ holds. This means precisely that (B, b) is a commutative semigroup.

2.C. **Locally Finitely Presentable Categories and Dense Functors.** Locally finitely presentable categories were introduced by Peter Gabriel and Friedrich Ulmer in their book [GU]. This concept generalizes the useful property of the category Set: every set can be reconstructed by knowing its finite subsets. It turns out that a set M can be recognized as finite when its hom-functor $\mathsf{Set}(M, -) : \mathsf{Set} \longrightarrow \mathsf{Set}$ preserves colimits of a certain class, called *filtered*.

A *filtered colimit* in general is a colimit of a functor $D : \mathcal{D} \longrightarrow \mathcal{K}$ where $\mathcal{D}$ is a small category that is *filtered*, i.e., such that every finite subcategory of $\mathcal{D}$ admits a cocone in $\mathcal{D}$. A functor preserving filtered colimits is called *finitary*. A monad is called *finitary* if its functor is finitary.

If $\mathcal{K}$ has filtered colimits, then an object M is called *finitely presentable* if its hom-functor $\mathcal{K}(M, -) : \mathcal{K} \longrightarrow \mathsf{Set}$ is finitary.

Example 2.3.

(1) A set is finitely presentable if and only if it is finite.
(2) An algebra of a finitary variety is finitely presentable if and only if it is finitely presentable in the ordinary sense of universal algebra, i.e., if it is presented by finitely many finitary equations and finitely many generators.

A general functor $F : \mathcal{D} \longrightarrow \mathcal{K}$ is called *dense* if its "tilde-conjugate" $\widetilde{F} : \mathcal{K} \longrightarrow [\mathcal{D}^{op}, \mathsf{Set}]$ defined by

$$\widetilde{F} : X \mapsto \mathcal{K}(F-, X)$$

is fully faithful. We will use the concept of density even when the category $\mathcal{D}$ is not small, since we will assume that the possibly illegitimate presheaf category $[\mathcal{D}^{op}, \mathsf{Set}]$ exists in some higher universe.

Definition 2.4. A cocomplete category with a small dense subcategory consisting of finitely presentable objects is called *locally finitely presentable* (*l.f.p.* for short).

Remark 2.5. L.f.p. categories are exactly the *essential algebraic* categories over sets, see, e.g., Chapter 3.D of [AR]. This means, roughly speaking, that l.f.p. categories encompass all categories of structures that are defined using (possibly partial) finitary operations. See Example 2.8 for some instances of this fact.

Notation 2.6. In what follows, $\mathcal{K}$ will always denote an l.f.p. category and $E : \mathcal{A} \longrightarrow \mathcal{K}$ will denote the full embedding of a small dense subcategory representing *all* finitely presentable objects. Objects of $\mathcal{A}$ will be denoted by small letters n, m, etc.

It can be proved that $E : \mathcal{A} \longrightarrow \mathcal{K}$ is in fact a free cocompletion of $\mathcal{A}$ under filtered colimits.

Example 2.7. Suppose $\mathbb{S}$ is a finitary monad on $\mathcal{K}$. The inclusion

$$A_{\mathbb{S}} : \mathcal{A}_{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{S}}$$

of a full subcategory spanned by $\mathbb{S}$-algebras free on objects of $\mathcal{A}$ is a dense category. This is proved, e.g., in Theorem 6.9 in [Bi].

Similarly, the inclusion

$$K_{\mathbb{S}} : \mathcal{K}_{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{S}}$$

of the Kleisli category in $\mathcal{K}^{\mathbb{S}}$ is dense, see Example 4.3 of [D].

Example 2.8.

(1) The category Set is l.f.p. As a small dense subcategory representing all finitely presentable objects we choose the category spanned by the sets $n = \{0, 1, \dots, n-1\}$, where n is a natural number.

(2) A poset, considered as a category, is l.f.p. if and only if it is an *algebraic lattice*. Finitely presentable objects are called *compact elements* in this context.

(3) Any finitary variety of universal algebras is l.f.p. As a small dense subcategory representing finitely presentable objects we can choose algebras, finitely presentable in the usual sense.

(4) The category Pos of all posets and monotone maps is l.f.p. As a small dense category representing finitely presentable objects one can choose the category of finite posets. The category Pos is not a finitary variety but it is an example of an essentially algebraic category defined by partial operations, see Chapter 3.D of [AR].

(5) The category Cat of all small categories and functors is l.f.p. As a small dense subcategory representing finitely presentable objects we can choose categories on finitely many objects subject to a finite set of commutativity conditions.

(6) If $\mathcal{K}$ is l.f.p., so is every functor category $[\mathcal{D}, \mathcal{K}]$, where $\mathcal{D}$ is a small category. Thus, all categories $[\mathcal{D}^{op}, \mathsf{Set}]$ of presheaves are l.f.p.

    Especially, the category

$$\mathsf{Fin}(\mathcal{K}, \mathcal{K})$$

    of *finitary endofunctors* of an l.f.p. category $\mathcal{K}$ is l.f.p., since $\mathsf{Fin}(\mathcal{K}, \mathcal{K})$ is equivalent to the functor category $[\mathcal{A}, \mathcal{K}]$, because $E : \mathcal{A} \longrightarrow \mathcal{K}$ is a free cocompletion under filtered colimits.

(7) Given a finitary monad $\mathbb{S}$ on an l.f.p. category $\mathcal{K}$, then the category $\mathcal{K}^{\mathbb{S}}$ is l.f.p. Recall from Example 2.7 that the inclusion $A_{\mathbb{S}} : \mathcal{A}_{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{S}}$ is dense and $\mathcal{A}_{\mathbb{S}}$ clearly consists of finitely presentable objects in $\mathcal{K}^{\mathbb{S}}$. Moreover, the category $\mathcal{K}^{\mathbb{S}}$ is cocomplete, being reflective in $[\mathcal{A}_{\mathbb{S}}^{op}, \mathsf{Set}]$. See, e.g., Theorem 6.9 in [Bi].

    Especially, the category

$$\mathsf{Mnd}_{\mathit{fin}}(\mathcal{K})$$

    of all finitary monads on an l.f.p. category $\mathcal{K}$ and monad morphisms is l.f.p. This is seen as follows:

    (a) The obvious finitary forgetful functor

$$U : \mathsf{Mnd}_{\mathit{fin}}(\mathcal{K}) \longrightarrow \mathsf{Fin}(\mathcal{K}, \mathcal{K})$$

    has a left adjoint given by a *free monad* F_H on a finitary endofunctor H.

    (b) Moreover, the functor U is *monadic*. This means that there exists a (finitary!) monad $\mathbb{S}$ on $\mathsf{Fin}(\mathcal{K}, \mathcal{K})$ such that the category $\mathsf{Mnd}_{\mathit{fin}}(\mathcal{K})$ is canonically equivalent to the category of $\mathbb{S}$-algebras. (This fact is easily proved using *Beck's Theorem*, see, e.g., Theorem 21.5.7 of [S].)

    (c) Now use the fact that $\mathsf{Fin}(\mathcal{K}, \mathcal{K})$ is l.f.p., hence $\mathsf{Fin}(\mathcal{K}, \mathcal{K})^{\mathbb{S}}$ (that is, the category of finitary monads on $\mathcal{K}$) is l.f.p.

2.D. **The Calculus of Finitary Monads.** The calculus of finitary monads on l.f.p. categories was developed by Max Kelly and John Power in their paper [KP]. This beautiful and powerful technique allows one to say that finitary monads "are" indeed finitary algebraic theories on an l.f.p. category, i.e., that finitary monads can be presented by equations and operations of a suitable finitary signature. We recall the basic ideas now, all the details and results can be found in the paper cited above. We will heavily use the equivalence

$$\mathsf{Fin}(\mathcal{K}, \mathcal{K}) \simeq [\mathcal{A}, \mathcal{K}]$$

see Example 2.8(6).

A *finitary signature* Σ is a collection

$$(\Sigma(n))_n$$

of objects of $\mathcal{K}$ indexed by finitely presentable objects of $\mathcal{K}$ and $\Sigma(n)$ is the object of *n-ary operations*.

Every finitary signature Σ gives rise to a corresponding *finitary polynomial endofunctor* H_Σ given by

$$H_\Sigma : X \mapsto \coprod_n \mathcal{K}(En, X) \bullet \Sigma(n)$$

where $\mathcal{K}(En, X) \bullet \Sigma(n)$ denotes the $\mathcal{K}(En, X)$-fold copower of $\Sigma(n)$ and where the coproduct is taken over all finitely presentable objects in $\mathcal{A}$. Intuitively, $H_\Sigma X$ is the object of *flat* Σ*-terms*, i.e., Σ-terms of depth ≤ 1.

A Σ*-algebra* is an algebra for H_Σ, i.e., a pair (A, a), where $a : H_\Sigma A \longrightarrow A$ is a morphism in $\mathcal{K}$ (intuitively: the computation of flat Σ-terms in A). A *homomorphism of* Σ*-algebras* from (A, a) to (B, b) is a morphism $h : A \longrightarrow B$ making the square

$$\begin{tikzcd} H_\Sigma A \arrow[r, "H_\Sigma h"] \arrow[d, "a"'] & H_\Sigma B \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

commutative.

If we denote by $\mathbb{F}_\Sigma = (F_\Sigma, \eta^\Sigma, \mu^\Sigma)$ the free monad on H_Σ (see Example 2.8(7)), it is straightforward to see that the category H_Σ-Alg of H_Σ-algebras and their homomorphisms is equivalent to the Eilenberg-Moore category $\mathcal{K}^{\mathbb{F}_\Sigma}$.

Define the *category of finitary signatures on* $\mathcal{K}$ *and their homomorphisms* as a functor category $[|\mathcal{A}|, \mathcal{K}]$ (where $|\mathcal{A}|$ is a discrete category on objects of $\mathcal{A}$) and denote it by

$$\mathsf{Sig}(\mathcal{K})$$

Then we have a series of right adjoints

$$\mathsf{Mnd}_{\mathit{fin}}(\mathcal{K}) \xrightarrow{U} \mathsf{Fin}(\mathcal{K}, \mathcal{K}) \xrightarrow{V} \mathsf{Sig}(\mathcal{K})$$

where the left adjoint of U forms a free monad on a finitary endofunctor and the left adjoint of V forms a polynomial endofunctor of the given finitary signature.

Steve Lack proved in [L] that the composite $V \cdot U : \mathsf{Mnd}_{\mathit{fin}}(\mathcal{K}) \longrightarrow \mathsf{Sig}(\mathcal{A})$ is monadic (compare with Example2.8(7)), yielding thus a canonical coequalizer presentation

$$\mathbb{F}_\Gamma \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \mathbb{F}_\Sigma \xrightarrow{c_{\mathbb{S}}} \mathbb{S} \tag{2.8}$$

for every finitary monad $\mathbb{S}$ on $\mathcal{K}$, where $\mathbb{F}_\Sigma$ and $\mathbb{F}_\Gamma$ are free monads on suitable signatures Σ and Γ.

What Max Kelly and John Power proved is that the category of Eilenberg-Moore alegbras $\mathcal{K}^{\mathbb{T}}$ is isomorphic to the full subcategory of $\mathbb{F}_\Sigma$-algebras (A, a) (or, equivalently, Σ-algebras) satisfying the equation $\lambda = \rho$. The idea is to replace algebras for a monad by certain monad morphisms (see Remark 2.11 for the well-known instance of these ideas). This is done as follows:

For every pair A, B of objects in $\mathcal{K}$ and every finitely presentable object n in $\mathcal{A}$ define

$$\langle\langle A, B\rangle\rangle n = \mathcal{K}(En, A) \pitchfork B$$

where $\mathcal{K}(En, A) \pitchfork B$ is the $\mathcal{K}(En, A)$-fold power of B in $\mathcal{K}$. Then the assignment $n \mapsto \langle\langle A, B\rangle\rangle n$ clearly extends to a functor $\mathcal{A} \longrightarrow \mathcal{K}$ thus $\langle\langle A, B\rangle\rangle$ is a finitary endofunctor of $\mathcal{K}$ if we identify $\mathsf{Fin}(\mathcal{K}, \mathcal{K})$ with the functor category $[\mathcal{A}, \mathcal{K}]$. Clearly, the definition of $\langle\langle A, B\rangle\rangle$ is functorial contravariantly in A and covariantly in B. Moreover, the following holds:

Proposition 2.9. *Every functor* $\langle\langle A, A\rangle\rangle$ *has a natural structure of a finitary monad and monad morphisms* $\breve{a} : \mathbb{T} \longrightarrow \langle\langle A, A\rangle\rangle$ *correspond uniquely to Eilenberg-Moore algebra structures* $a : TA \longrightarrow A$ *on* A*. Moreover, if* $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ *is a monad morphism, then the equality*

$$(a \cdot \alpha_A)\breve{} = \breve{a} \cdot \alpha \tag{2.9}$$

holds.

If, for $f : A \longrightarrow B$, we form the pullback

$$\begin{tikzcd} \{\{f, f\}\} \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & \langle\langle A, A\rangle\rangle \arrow[d, "\langle\langle A, f\rangle\rangle"] \\ \langle\langle B, B\rangle\rangle \arrow[r, "\langle\langle f, B\rangle\rangle"'] & \langle\langle A, B\rangle\rangle \end{tikzcd} \tag{2.10}$$

in $\mathsf{Fin}(\mathcal{K}, \mathcal{K})$ then the following holds:

Proposition 2.10. *$\{\{f,f\}\}$ has canonically a structure of a finitary monad and there exists a monad morphism $\mathbb{T} \longrightarrow \{\{f,f\}\}$ precisely when f carries a $\mathbb{T}$-algebra homomorphism between the corresponding $\mathbb{T}$-algebras.*

Consider now the diagram

$$\begin{tikzcd} \mathbb{F}_\Gamma \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & \mathbb{F}_\Sigma \arrow[r, "c_{\mathbb{S}}"] \arrow[dr] & \mathbb{S} \arrow[d] \\ & & \langle\langle A, A\rangle\rangle \end{tikzcd}$$

to conclude that monad morphisms $\mathbb{S} \longrightarrow \langle\langle A,A\rangle\rangle$ (i.e., $\mathbb{S}$-algebras) correspond bijectively to monad morphisms $\mathbb{F}_\Sigma \longrightarrow \langle\langle A,A\rangle\rangle$ that coequalize the pair λ, ρ (i.e., to the Σ-algebras that satisfy the equations λ, ρ). Replacing $\langle\langle A,A\rangle\rangle$ by $\{\{f,f\}\}$ in the above diagram one obtains the corresponding bijection of hom-sets, proving that the category of $\mathbb{S}$-algebras and their homomorphisms is equivalent to the category of Σ-algebras that satisfy $\lambda = \rho$ and their homomorphisms.

Remark 2.11. In fact, the above calculus of monads is a generalization of a well known fact about *actions of monoids*: denote, for sets A, B, by $\langle\langle A,B\rangle\rangle$ the set of all maps from A to B.

This definition is clearly functorial contravariantly in A and covariantly in B. Moreover, it is well-known that every set $\langle\langle A,A\rangle\rangle$ carries a natural structure of a monoid w.r.t. composition. Furthermore, given any monoid $\mathbb{T} = (T, e, *)$, there is an evident bijection between monoid homomorphisms from $\mathbb{T}$ to $\langle\langle A,A\rangle\rangle$ and monoid actions $T \times A \longrightarrow A$ (compare with Proposition 2.9).

If we define, for a map $f : A \longrightarrow B$, the set $\{\{f,f\}\}$ as a pullback

$$\begin{tikzcd} \{\{f,f\}\} \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & \langle\langle A,A\rangle\rangle \arrow[d, "\langle\langle A,f\rangle\rangle"] \\ \langle\langle B,B\rangle\rangle \arrow[r, "\langle\langle f,B\rangle\rangle"'] & \langle\langle A,B\rangle\rangle \end{tikzcd}$$

i.e., elements of $\{\{f,f\}\}$ are pairs $(h : A \longrightarrow A, k : B \longrightarrow B)$ such that $f \cdot h = k \cdot f$, then $\{\{f,f\}\}$ becomes a monoid in a natural way. In fact, $\{\{f,f\}\}$ is a submonoid of a product $\langle\langle A,A\rangle\rangle \times \langle\langle B,B\rangle\rangle$ via the map $\langle \pi_A, \pi_B\rangle$ (we will use a generalization of this fact in Corollary 5.5 below).

Moreover, to give a monoid homomorphisms from a monoid $\mathbb{T}$ to $\{\{f,f\}\}$ is to say that $f : A \longrightarrow B$ is an equivariant map between the action of $\mathbb{T}$ on A and B, respectively (compare with Proposition 2.10).

3. Sufficient Conditions for Density

Assumption 3.1. In the rest of the paper $\mathcal{K}$ denotes a fixed l.f.p. category, $E : \mathcal{A} \longrightarrow \mathcal{K}$ the inclusion of a small dense subcategory representing all finitely presentable objects of $\mathcal{K}$. By $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ we always denote a morphism of finitary monads on $\mathcal{K}$.

Definition 3.2. A monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ is called *dense* if the restriction-along-α functor $\alpha_* : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}^{\mathbb{S}}$ is fully faithful.

Since $\mathcal{K}$ (and hence $\mathcal{K}^{\mathbb{T}}$, see Example 2.8(7)) is cocomplete, the functor α_* always has a left adjoint α^* (see (2.7)) and by a general result on adjunctions, α_* is fully faithful if and only if α^* is dense (see, e.g., Proposition 17.2.6 of [S]). This motivated our choice of terminology.

In this section we mention various sufficient conditions for density of α that are mostly well-known and are often very easy to verify in practice. Most of the properties suggest that density of α is a kind of "epimorphism" condition, as, for example, the following trivial property:

Lemma 3.3. *Dense monad morphisms compose and if the composite $\beta \cdot \alpha$ is dense, so is β.*

Lemma 3.4. *Every pointwise epimorphic monad morphism is dense.*

Proof. Consider the diagram

$$\begin{tikzcd} SA \arrow[r, "Sh"] \arrow[d, "\alpha_A"'] & SB \arrow[d, "\alpha_B"] \\ TA \arrow[r, "Th"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

where (A, a) and (B, b) are arbitrary $\mathbb{T}$-algebras. If the perimeter of the above diagram commutes, so does the lower square since α_A is epi. □

Remark 3.5. In fact, pointwise epimorphic α characterize abstractly *Birkhoff subcategories* of $\mathcal{K}^{\mathbb{S}}$, see Theorem 3.3.4 of [M]. A Birkhoff subcategory of $\mathcal{K}^{\mathbb{S}}$ is one where we add equations to the presentation of $\mathbb{S}$, see the abovementioned theorem in [M]. An example of a Birkhoff subcategory in groups is the subcategory of Abelian groups.

However, the full inclusion $\alpha_* : \mathsf{Groups} \longrightarrow \mathsf{Monoids}$ (see Example 1.1(2)) is an example of a dense α that is *not* pointwise epimorphic.

Observe that by (2.5) every α_A is an $\mathbb{S}$-homomorphism from (SA, μ_A^S) to $\alpha_*(TA, \mu_A^T)$.

Lemma 3.6. *Suppose $\alpha_m : (Sm, \mu_m^S) \longrightarrow \alpha_*(Tm, \mu_m^T)$ is an epimorphism in $\mathcal{K}^{\mathbb{S}}$ for every finitely presentable object m. Then α is dense.*

Proof. We prove first that that the action of α_* is a bijection on every hom-set of the form $\mathcal{K}^{\mathbb{T}}((Tm, \mu_m^T), (B, b))$, where m is finitely presentable and (B, b) is an arbitrary $\mathbb{T}$-algebra.

Let $h : \alpha_*(Tm, \mu_m^T) \longrightarrow \alpha_*(B, b)$ be any $\mathbb{S}$-homomorphism and denote by $h' : (Tm, \mu_m^T) \longrightarrow (B, b)$ the unique $\mathbb{T}$-homomorphism extending the composite $h \cdot \eta_m^T : m \longrightarrow B$. We want to prove that $\alpha_*(h') = h$. But this follows from the fact that the diagram

$$\begin{tikzcd} (Sm, \mu_m^S) \arrow[r, "\alpha_m"] & \alpha_*(Tm, \mu_m^T) \arrow[r, shift left, "h"] \arrow[r, shift right, "\alpha_*(h')"'] & \alpha_*(B, b) \end{tikzcd}$$

clearly commutes and from the assumption that α_m is an epimorphism in $\mathcal{K}^{\mathbb{S}}$.

Expressing every free $\mathbb{T}$-algebra (TA, μ_A^T) as a filtered colimit

$$Tf : (Tm, \mu_m^T) \longrightarrow (TA, \mu_A^T)$$

where $f : m \longrightarrow A$, we conclude that the action of α_* is a bijection on every hom-set of the form $\mathcal{K}^{\mathbb{T}}((TA, \mu_A^T), (B, b))$, where A is arbitrary and (B, b) is an arbitrary $\mathbb{T}$-algebra.

Finally, let us choose any $\mathbb{S}$-homomorphism $h : \alpha_*(A, a) \longrightarrow \alpha_*(B, b)$. Express (A, a) as a canonical coequalizer

$$\begin{tikzcd} (TTA, \mu_{TA}^T) \arrow[r, shift left, "\mu_A^T"] \arrow[r, shift right, "Ta"'] & (TA, \mu_A^T) \arrow[r, "a"] & (A, a) \end{tikzcd}$$

in $\mathcal{K}^{\mathbb{T}}$ and consider the following commutative diagram

$$\begin{tikzcd} \alpha_*(TTA, \mu_{TA}^T) \arrow[r, shift left, "\alpha_*(\mu_A^T)"] \arrow[r, shift right, "\alpha_*(Ta)"'] & \alpha_*(TA, \mu_A^T) \arrow[r, "\alpha_*(a)"] \arrow[dr, dotted, "\alpha_*(k')"'] & \alpha_*(A, a) \arrow[d, "h"] \\ & & \alpha_*(B, b) \end{tikzcd}$$

We know that the composite $h \cdot \alpha_*(a)$ is of the form $\alpha_*(k')$ for some $\mathbb{T}$-homomorphism $k' : (TA, \mu_A^T) \longrightarrow (B, b)$ necessarily coequalizing the pair $Ta, \mu_A^T : (TTA, \mu_{TA}^T) \longrightarrow (TA, \mu_A^T)$. Hence k' induces a unique $\mathbb{T}$-homomorphism $h' : (A, a) \longrightarrow (B, b)$ for which $\alpha_*(h') = h$ holds. □

Example 3.7. The above lemma can be applied to the case of $\alpha_* : \mathsf{Groups} \longrightarrow \mathsf{Monoids}$, see Example 1.1(2), since the inclusion of a free monoid into a free group on the same (finite) set is an epimorphism of monoids.

Remark 3.8. It should be noted that the condition on α_m of Lemma 3.6 can be easily modified into a necessary and sufficient condition for density of α:

Say that $\alpha_m : (Sm, \mu^S_m) \longrightarrow \alpha_*(Tm, \mu^T_m)$ is an *α_*-epimorphism* in $\mathcal{K}^{\mathbb{S}}$ if from $h \cdot \alpha_m = k \cdot \alpha_m$ the equality $h = k$ follows for any parallel pair $h, k : \alpha_*(Tm, \mu^T_m) \longrightarrow \alpha_*(B, b)$ of $\mathbb{S}$-homomorphisms.

Then α is dense if and only if every α_m is α_*-epimorphism in $\mathcal{K}^{\mathbb{S}}$.

For sufficiency, read the proof of Lemma 3.6. Conversely, if α is dense and if $h \cdot \alpha_m = k \cdot \alpha_m$ holds for a parallel pair $h, k : \alpha_*(Tm, \mu^T_m) \longrightarrow \alpha_*(B, b)$ of $\mathbb{S}$-homomorphisms, then $h = \alpha_*(h')$ and $k = \alpha_*(k')$ for a unique parallel pair $h', k' : (Tm, \mu^T_m) \longrightarrow (B, b)$ of $\mathbb{T}$-homomorphisms. Observe that $h' = k'$ holds, since both h' and k' extend the same morphism:

$$h' \cdot \eta^T_m = h \cdot \alpha_m \cdot \eta^S_m = k \cdot \alpha_m \cdot \eta^S_m = k' \cdot \eta^T_m$$

We conclude that $h = k$.

Lemma 3.9. *Every regular epimorphism in* $\mathsf{Mnd}_{fin}(\mathcal{K})$ (*especially, every split epimorphism*) *is dense.*

Proof. Consider a coequalizer

$$\mathbb{P} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{S} \xrightarrow{\alpha} \mathbb{T}$$

in $\mathsf{Mnd}_{fin}(\mathcal{K})$. This coequalizer gives rise to an equalizer

$$\mathcal{K}^{\mathbb{T}} \xrightarrow{\alpha_*} \mathcal{K}^{\mathbb{S}} \underset{\rho_*}{\overset{\lambda_*}{\rightrightarrows}} \mathcal{K}^{\mathbb{P}}$$

of functors that commute with (faithful) forgetful functors. Thus, α_* is fully faithful. $\square$

Recall that a monad $\mathbb{S}$ is called *idempotent* if the underlying functor $U^{\mathbb{S}} : \mathcal{K}^{\mathbb{S}} \longrightarrow \mathcal{K}$ is fully faithful.

Lemma 3.10. *A monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ with $\mathbb{S}$ idempotent is a dense morphism if and only if $\mathbb{T}$ is idempotent.*

Proof. The statement follows from the fact that $U^{\mathbb{S}} \cdot \alpha_* = U^{\mathbb{T}}$ holds, hence $U^{\mathbb{T}}$ is fully faithful if and only if α_* is fully faithful. $\square$

Recall that by $A_{\mathbb{S}} : \mathcal{A}_{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{S}}$, $A_{\mathbb{T}} : \mathcal{A}_{\mathbb{T}} \longrightarrow \mathcal{K}^{\mathbb{T}}$, resp., we denote the full dense subcategories of $\mathbb{S}$-algebras, $\mathbb{T}$-algebras, resp., spanned by algebras free on finitely presentable objects (see Example 2.7). Every monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ then induces a functor

$$A_\alpha : \mathcal{A}_{\mathbb{S}} \longrightarrow \mathcal{A}_{\mathbb{T}} \tag{3.1}$$

sending every $\mathbb{S}$-algebra (Sn, μ^S_n) to (Tn, μ^T_n) and every $\mathbb{S}$-homomorphism $h : (Sn, \mu^S_n) \longrightarrow (Sm, \mu^S_m)$ to a $\mathbb{T}$-homomorphism $(\alpha_m \cdot h \cdot \eta^S_n)^* : (Tn, \mu^T_n) \longrightarrow (Tm, \mu^T_m)$ where by upper star we denote the unique extension to a $\mathbb{T}$-homomorphism.

Clearly, the above process can be performed for arbitrary free algebras, yielding a functor

$$K_\alpha : \mathcal{K}_{\mathbb{T}} \longrightarrow \mathcal{K}_{\mathbb{S}} \tag{3.2}$$

Lemma 3.11. *If the functor $[A^{op}_\alpha, \mathsf{Set}] : [\mathcal{A}^{op}_{\mathbb{T}}, \mathsf{Set}] \longrightarrow [\mathcal{A}^{op}_{\mathbb{S}}, \mathsf{Set}]$ is fully faithful, then α is a dense monad morphism.*

Proof. The square

$$\begin{tikzcd} \mathcal{K}^{\mathbb{T}} \arrow[r, "\widetilde{A_{\mathbb{T}}}"] \arrow[d, "\alpha_*"'] & {[\mathcal{A}^{op}_{\mathbb{T}}, \mathsf{Set}]} \arrow[d, "{[A^{op}_\alpha, \mathsf{Set}]}"] \\ \mathcal{K}^{\mathbb{S}} \arrow[r, "\widetilde{A_{\mathbb{S}}}"'] & {[\mathcal{A}^{op}_{\mathbb{S}}, \mathsf{Set}]} \end{tikzcd} \tag{3.3}$$

is easily seen to commute (up to isomorphism). Its diagonal is fully faithful by assumption, thus, the functor α_* is fully faithful. $\square$

Remark 3.12. Functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ between general small categories with $[F^{op}, \mathsf{Set}]$ fully faithful are called *connected* and they were characterized in [ABSV] as those for which the canonical morphism

$$\int^C \mathcal{D}(D', FC) \times \mathcal{D}(FC, D) \longrightarrow \mathcal{D}(D', D)$$

given by composition is an isomorphism.

4. Beth's Definability Theorem

In this section we characterize dense monad morphisms in "Beth style" and give some connections between dense monad morphisms and the density of induced functors between Kleisli categories.

The following concept captures the notion of "explicit definability".

Definition 4.1. We say that an n-tuple $\tau : n \longrightarrow Tm$ of m-ary $\mathbb{T}$-operations is $\mathbb{S}$-*definable*, if there exists an equation of the form

$$m + n + p \overset{\lambda_\tau}{\underset{\rho_\tau}{\rightrightarrows}} S(m + n + q)$$

with p, q finitely presentable, such that

$$\alpha_*(A, a) \models \forall x \forall y \left(y = \tau(x) \Leftrightarrow \exists t \left(\lambda_\tau(x, y, t) = \rho_\tau(x, y, t)\right)\right)$$

holds for every $\mathbb{T}$-algebra (A, a).

Example 4.2.

(1) As mentioned in the introduction, inverses in groups are definable in the language of monoids. More precisely, every group $(G, *, e, (-)^{-1})$ satisfies the sentence

$$\forall x \forall y \left(y = x^{-1} \Leftrightarrow x * y = e \wedge y * x = e\right)$$

Thus, if $\mathbb{S}$ and $\mathbb{T}$ are the finitary monads on Set that correspond to the theory of monoids and groups, respectively, then $(-)^{-1} : 1 \longrightarrow T1$ is $\mathbb{S}$-definable.

(2) Complements in Boolean algebras are definable in the language of distributive lattices having the least and the greatest elements: every Boolean algebra $(B, \sqcap, \sqcup, \bot, \top, \neg(-))$ satisfies the sentence

$$\forall x \forall y \left(y = \neg x \Leftrightarrow (x \sqcap y = \bot \wedge x \sqcup y = \top)\right)$$

(3) Greatest common divisors in Euclidean rings are definable in the language of rings: every Euclidean ring $(R, +, \cdot, 0, 1, \gcd(-, -))$ satisfies the sentence

$$\forall x \forall x' \forall y \left(y = \gcd(x, x') \Leftrightarrow \exists k \exists k' \exists a \exists a' (x = k \cdot y \wedge x' = k' \cdot y \wedge y = a \cdot x + a' \cdot x')\right)$$

(4) Joins and meets in Boolean algebras are definable in the language of MV-algebras (see [C]): every Boolean algebra $(B, \sqcap, \sqcup, \bot, \top, \neg(-))$ satisfies the following two sentences

$$\forall x \forall x' \forall y \left(y = x \sqcup x' \Leftrightarrow y = \neg(\neg x \oplus x') \oplus x'\right)$$
$$\forall x \forall x' \forall y \left(y = x \sqcap x' \Leftrightarrow y = \neg(\neg x \sqcup \neg x')\right)$$

(5) Every n-tuple $\sigma : n \longrightarrow Sm$ of m-ary $\mathbb{S}$-operations is $\mathbb{S}$-definable. Consider the equation $\lambda_\sigma, \rho_\sigma : n \longrightarrow S(m + n)$ with

$$\lambda_\sigma \equiv n \xrightarrow{\sigma} Sm \xrightarrow{S\mathrm{inl}} S(m + n)$$

and

$$\rho_\sigma \equiv n \xrightarrow{\eta_n^S} Sn \xrightarrow{S\mathrm{inr}} S(m + n)$$

Then every $\mathbb{S}$-algebra of the form $\alpha_*(A, a)$ (indeed, *every* $\mathbb{S}$-algebra) satisfies the sentence

$$\forall x \forall y \left(y = \sigma(x) \Leftrightarrow \lambda_\sigma(x, y) = \rho_\sigma(x, y)\right)$$

as it is easily seen.

The concept of $\mathbb{S}$-definability allows us to characterize dense monad morphisms:

Theorem 4.3. *The following are equivalent:*

(1) $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ *is a dense monad morphism.*
(2) *Every* $\mathbb{S}$*-homomorphism* $\alpha_*(Tk, \mu_k^T) \longrightarrow \alpha_*(A, a)$*, where* k *is finitely presentable, is a* $\mathbb{T}$*-homomorphism* $(Tk, \mu_k^T) \longrightarrow (A, a)$*.*
(3) *Every* n*-tuple of* m*-ary* $\mathbb{T}$*-operations is* $\mathbb{S}$*-definable.*

Proof. That (1) implies (2) and (3) implies (1) is trivial. It remains to prove that (2) implies (3): Suppose an n-tuple $\tau : n \longrightarrow Tm$ of m-ary $\mathbb{T}$-operations is given. The plan of the proof is as follows: we exhibit first a "large" $\mathbb{S}$-equation L_τ, R_τ, that will witness $\mathbb{S}$-definability of τ and then we reduce it to a "small" $\mathbb{S}$-equation λ_τ, ρ_τ, using finitarity of $\mathbb{S}$ and $\mathbb{T}$.

Define first an equation

$$m + n + STm \underset{R_\tau = [R^{(i)}, R^{(ii)}, R^{(iii)}]}{\overset{L_\tau = [L^{(i)}, L^{(ii)}, L^{(iii)}]}{\rightrightarrows}} S(m + n + Tm)$$

with the following properties (where (A, a) is an arbitrary $\mathbb{T}$-algebra):

(i) $\alpha_*(A, a) \models L^{(i)}(x, y, t) = R^{(i)}(x, y, t)$ if and only if $t \cdot \eta^T_m = x$.
(ii) $\alpha_*(A, a) \models L^{(ii)}(x, y, t) = R^{(ii)}(x, y, t)$ if and only if $t \cdot \tau = y$.
(iii) $\alpha_*(A, a) \models L^{(iii)}(x, y, t) = R^{(iii)}(x, y, t)$ if and only if t is an $\mathbb{S}$-homomorphism from $\alpha_*(Tm, \mu^T_m)$ to $\alpha_*(A, a)$.

The individual equations are defined as follows:

(i) Put

$$L^{(i)} \equiv m \xrightarrow{\eta^S_m} Sm \xrightarrow{S\mathrm{inl}} S(m + n + Tm)$$
$$R^{(i)} \equiv m \xrightarrow{\eta^T_m} Tm \xrightarrow{\mathrm{inr}} m + n + Tm \xrightarrow{\eta^S_{m+n+Tm}} S(m + n + Tm)$$

Let (A, a) be any $\mathbb{T}$-algebra and let $[x, y, t] : m+n+Tm \longrightarrow A$ be any morphism. Then the equations

$$[x, y, t]^\sharp \cdot L^{(i)} = [x, y, t]^\sharp \cdot S\mathrm{inl} \cdot \eta^S_m = x^\sharp \cdot \eta^S_m = x$$

and

$$[x, y, t]^\sharp \cdot R^{(i)} = [x, y, t]^\sharp \cdot \eta^S_{m+n+Tm} \cdot \mathrm{inr} \cdot \eta^T_m = [x, y, t] \cdot \mathrm{inr} \cdot \eta^T_m = t \cdot \eta^T_m$$

hold, thus, $\alpha_*(A, a) \models L^{(i)}(x, y, t) = R^{(i)}(x, y, t)$ if and only if $t \cdot \eta^T_m = x$.

(ii) Put

$$L^{(ii)} \equiv n \xrightarrow{\mathrm{inm}} m + n + Tm \xrightarrow{\eta^S_{m+n+Tm}} S(m + n + Tm)$$
$$R^{(ii)} \equiv n \xrightarrow{\tau} Tm \xrightarrow{\mathrm{inr}} m + n + Tm \xrightarrow{\eta^S_{m+n+Tm}} S(m + n + Tm)$$

Let (A, a) be any $\mathbb{T}$-algebra and let $[x, y, t] : m+n+Tm \longrightarrow A$ be any morphism. Then the equations

$$[x, y, t]^\sharp \cdot L^{(ii)} = [x, y, t]^\sharp \cdot \eta^S_{m+n+Tm} \cdot \mathrm{inm} = [x, y, t] \cdot \mathrm{inm} = y$$

and

$$[x, y, t]^\sharp \cdot R^{(ii)} = [x, y, t]^\sharp \cdot \eta^S_{m+n+Tm} \cdot \cdot \mathrm{inr} \cdot \tau = [x, y, t] \cdot \mathrm{inr} \cdot \tau = t \cdot \tau$$

hold, thus, $\alpha_*(A, a) \models L^{(ii)}(x, y, t) = R^{(ii)}(x, y, t)$ if and only if $t \cdot \tau = y$.

(iii) Put

$$L^{(iii)} \equiv STm \xrightarrow{\alpha_{Tm}} TTm \xrightarrow{\mu^T_m} Tm \xrightarrow{\mathrm{inr}} m + n + Tm \xrightarrow{\eta^S_{m+n+Tm}} S(m + n + Tm)$$
$$R^{(iii)} \equiv STm \xrightarrow{S\mathrm{inr}} S(m + n + Tm)$$

Then the equations

$$[x, y, t]^\sharp \cdot L^{(iii)} = [x, y, t]^\sharp \cdot \eta^S_{m+n+Tm} \cdot \mathrm{inr} \cdot \mu^T_m \cdot \alpha^T_{Tm} = [x, y, t] \cdot \mathrm{inr} \cdot \mu^T_m \cdot \alpha^T_{Tm} = t \cdot \mu^T_m \cdot \alpha^T_{Tm}$$

and

$$[x, y, t]^\sharp \cdot R^{(iii)} = [x, y, t]^\sharp \cdot S\mathrm{inr} = a \cdot \alpha_A \cdot St$$

hold, thus, $\alpha_*(A, a) \models L^{(iii)}(x, y, t) = R^{(iii)}(x, y, t)$ if and only if t is an $\mathbb{S}$-homomorphism from $\alpha_*(Tm, \mu^T_m)$ to $\alpha_*(A, a)$.

We proved so far that $\alpha_*(A, a) \models L_\tau(x, y, t) = R_\tau(x, y, t)$ if and only if t is the unique $\mathbb{T}$-homomorphism from (Tm, μ^T_m) to (A, a) (since α_* is assumed to be bijective on homomorphisms from (Tm, μ^T_m) to (A, a)) that extends $x : m \longrightarrow A$ and satisfies $t \cdot \tau = y$ (or, equivalently, the equality $\tau(x) = y$ holds).

Suppose that x, y is given and $y = \tau(x)$ holds, then it suffices to put t to be the extension of x. If x, y are given such that $y \neq \tau(x)$, then there is no extension of x into a $\mathbb{T}$-homomorphism t since this would imply $y = \tau(x)$.

We now reduce the "large" equation L_τ, R_τ to a finitary one. Consider all morphisms $f : p \longrightarrow STm$ with p finitely presentable and suppose that for every such f there exists a $\mathbb{T}$-algebra (A_f, a_f) and a triple $[x_f, y_f, t_f] : m + n + Tm \longrightarrow A_f$ such that $y_f \neq \tau(x_f)$ and the diagram

$$m+n+p \xrightarrow{\mathrm{id}+\mathrm{id}+f} m+n+STm \overset{L_\tau}{\underset{R_\tau}{\rightrightarrows}} S(m+n+Tm) \xrightarrow{[x_f,y_f,t_f]^\sharp} \alpha_*(A_f, a_f)$$

commutes. Since

$$\prod_f \alpha_*(A_f, a_f) \cong \alpha_*(\prod_f (A_f, a_f))$$

holds, this shows that the unique induced morphism

$$[x, y, t]^\sharp : S(m+n+Tm) \longrightarrow \alpha_*(\prod_f (A_f, a_f))$$

coequalizes L_τ, R_τ and $y \neq \tau(x)$ holds, a contradiction.

Therefore we may choose $f : p \longrightarrow STm$ with p finitely presentable such that the equation

$$m+n+p \xrightarrow{\mathrm{id}+\mathrm{id}+f} m+n+STm \overset{L_\tau}{\underset{R_\tau}{\rightrightarrows}} S(m+n+Tm) \tag{4.1}$$

does the same job as L_τ, R_τ. Express now $S(m+n+Tm)$ as a filtered colimit with the colimit cocone

$$S(\mathrm{id}+\mathrm{id}+g) : S(m+n+q) \longrightarrow S(m+n+Tm), \quad g : q \longrightarrow Tm \text{ with } q \text{ finitely presentable}$$

and use the fact that $m+n+p$ is finitely presentable to obtain a following factorization of (4.1)

$$\begin{tikzcd} m+n+p \arrow[r, "\mathrm{id}+\mathrm{id}+f"] \arrow[rr, bend left, "\lambda_\tau"] \arrow[rr, bend right, "\rho_\tau"'] & m+STm \arrow[r, shift left] \arrow[r, shift right] & S(m+n+q) \arrow[r, "S(\mathrm{id}+\mathrm{id}+g)"] & S(m+n+Tm) \end{tikzcd} \tag{4.2}$$

through a colimit. In this way we obtain the desired λ_τ, ρ_τ. Clearly, if $[x, y, t]^\sharp$ coequalizes (4.1) then $[x, y, t \cdot g]^\sharp$ coequalizes λ_τ, ρ_τ. Conversely, if $[x, y, t]^\sharp$ coequalizes λ_τ, ρ_τ, let $t^* : Tm \longrightarrow A$ be the unique extension of x to a $\mathbb{T}$-homomorphism. Then $[x, y, t^*]^\sharp$ coequalizes (4.1). ☐

Remark 4.4.

(1) The equivalence of (1) and (3) of the above theorem is indeed Beth's Definability Theorem for finitary monads: the implication (1) ⇒ (3) says that every *implicitly defined operation* (i.e., one preserved by $\mathbb{S}$-homomorphisms) is *defined explicitly* by a system of $\mathbb{S}$-equations. The implication (3) ⇒ (1) is then the trivial part of Beth's Definability Theorem: every explicitly defined operation is preserved by $\mathbb{S}$-homomorphisms.

(2) Let us stress that the proof of (2) ⇒ (3) in Theorem 4.3 is *non-constructive* which makes it, in general, difficult to find the definability $\mathbb{S}$-equation.

(3) One can prove the equivalence of (1) and (3) in a more general setting, imposing no restriction on $\mathcal{K}$, $\mathbb{S}$ and $\mathbb{T}$ whatsoever. One has to require $\mathbb{S}$-definability of every $\tau : X \longrightarrow TY$, where X and Y are arbitrary objects of $\mathcal{K}$. In proving (1) ⇒ (3) one defines an equation

$$Y + X + STY \overset{L_\tau}{\underset{R_\tau}{\rightrightarrows}} S(Y + X + TY)$$

in the same way as in the above proof. Of course, one cannot expect to reduce the above system to a "smaller" one.

(4) We show in Example 4.8 below that one cannot weaken the condition (2) of the above theorem to the requirement that α_* is fully faithful when restricted to $\mathcal{A}_\mathbb{T}$, the category of $\mathbb{T}$-algebras free on finitely presentable objects of $\mathcal{K}$.

Example 4.5. It is well-known (see [PS]) that the change-of-ring functor $\alpha_* : T\text{-}\mathsf{Mod} \longrightarrow S\text{-}\mathsf{Mod}$ between the categories of left modules is fully faithful if and only if $\alpha : S \longrightarrow T$ is an epimorphism of rings with a unit. This fits into our framework since every ring S with a unit can be considered as a finitary monad by assigning (an underlying set of) a free left S-module $S^{(X)}$ on X to every set X. Moreover, ring homomorphisms correspond then precisely to morphisms of the corresponding monads.

Observe that an n-tuple $\tau : n \longrightarrow T^{(m)}$ of m-ary $\mathbb{T}$ operations is just a matrix (τ_{ij}) of elements of T having n rows and m columns. We will denote this matrix by τ again.

Suppose A is any left T-module and $x : m \longrightarrow A$, $y : n \longrightarrow A$ are "vectors" of elements of A. Then $\tau(x) = y$ holds if and only if the system

$$\tau \cdot x = y$$

of linear equations holds in A.

Analogously, an $\mathbb{S}$-equation

$$m + n + p \overset{\lambda_\tau}{\underset{\rho_\tau}{\rightrightarrows}} S^{(m+n+q)}$$

consists of a pair λ_τ, ρ_τ of matrices over S having $(m + n + p)$ rows and $(m + n + q)$ columns.

Theorem 4.3 then gives us the following characterization of ring epimorphisms $\alpha : S \longrightarrow T$:

For every pair n, m of natural numbers and for every $(n \times m)$-matrix τ of elements of T there exist $((m + n + p) \times (m + n + q))$-matrices λ_τ, ρ_τ of elements of S, such that, for every left T-module A, and each pair x, y of vectors of A, the system

$$\tau \cdot x = y$$

of linear equations holds in A if and only if the system

$$\lambda_\tau \cdot \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \rho_\tau \cdot \begin{pmatrix} x \\ y \\ t \end{pmatrix}$$

has a solution t in A.

Example 4.6. Considerations similar to the previous example can be made to characterize epimorphims of monoids.

If we identify a monoid S with a one-object category, then the category S-Act of left S-actions and equivariant maps can be identified with the presheaf category $[S^{op}, \mathsf{Set}]$.

Given a monoid homomorphism $a : S \longrightarrow T$, then the restriction-along-a functor $[a^{op}, \mathsf{Set}] : [T^{op}, \mathsf{Set}] \longrightarrow [S^{op}, \mathsf{Set}]$ is fully faithful if and only if $a : S \longrightarrow T$ is an epimorphism of monoids, as proved in Example 3.13(3) of [BV].

Now every monoid S in sets defines a finitary monad $\mathbb{S}$ on the category of sets by putting $X \mapsto S \times X$ and, analogously, every monoid homomorphism $a : S \longrightarrow T$ defines a monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ having $\alpha_X = a \times X : S \times X \longrightarrow T \times X$ as components. Moreover, the restriction-along-a functor is of the form α_*, since the categories of left actions are precisely the categories of Eilenberg-Moore algebras for the respective monads. We conclude:

The monad morphism α is dense if and only if a is an epimorphism of monoids.

Identify every $\tau : n \longrightarrow T \times m$ with an n-tuple (τ_i, j_i). Suppose that A is equipped with a left T-action denoted by $\circ$. Then for $x : m \longrightarrow A$ and $y : n \longrightarrow A$ the equality $\tau(x) = y$ holds if and only if the system of equations

$$y_i = \tau_i \circ x_{j_i}, \quad i = 0, \ldots, n-1$$

holds in A. Now we can use Theorem 4.3 to characterize monoid epimorphisms in the style of Example 4.5.

Recall the functors $A_\alpha : \mathcal{A}_\mathbb{S} \longrightarrow \mathcal{A}_\mathbb{T}$ and $K_\alpha : \mathcal{K}_\mathbb{S} \longrightarrow \mathcal{K}_\mathbb{T}$ (see (3.1) and (3.2)).

Proposition 4.7. *The following are equivalent:*

(1) *α is a dense monad morphism.*
(2) *The composite $A_\mathbb{T} \cdot A_\alpha : \mathcal{A}_\mathbb{S} \longrightarrow \mathcal{K}^\mathbb{T}$ is a dense functor.*
(3) *The composite $K_\mathbb{T} \cdot K_\alpha : \mathcal{K}_\mathbb{S} \longrightarrow \mathcal{K}^\mathbb{T}$ is a dense functor.*

Proof. To prove (1) $\Leftrightarrow$ (2) observe that the diagonal of the square (3.3) is the functor $\widetilde{A_\mathbb{T} \cdot A_\alpha}$. It is fully faithful (i.e., $A_\mathbb{T} \cdot A_\alpha$ is dense) if and only if α_* is fully faithful, since the horizontal arrows of (3.3) are fully faithful (see Example 2.7).

The proof of (1) $\Leftrightarrow$ (3) is analogous: instead of (3.3) one uses the commutative square

$$\begin{tikzcd} \mathcal{K}^\mathbb{T} \arrow[r, "\widetilde{K_\mathbb{T}}"] \arrow[d, "\alpha_*"'] & {[\mathcal{K}_\mathbb{T}^{op}, \mathsf{Set}]} \arrow[d, "{[K_\alpha^{op}, \mathsf{Set}]}"] \\ \mathcal{K}^\mathbb{S} \arrow[r, "\widetilde{K_\mathbb{S}}"'] & {[\mathcal{K}_\mathbb{S}^{op}, \mathsf{Set}]} \end{tikzcd} \tag{4.3}$$

(which is even a bipullback, as proved by Fred Linton in [Li$_2$]). ☐

Example 4.8. We give an example of a functor $\alpha_* : \mathcal{K}^{\mathbb{R}} \longrightarrow \mathcal{K}^{\mathbb{F}}$ that is not fully faithful, although its restriction to $\mathcal{K}_{\mathbb{R}}$ is always fully faithful.

Given a finitary endofunctor $H : \mathcal{K} \longrightarrow \mathcal{K}$, consider the free monad $\mathbb{F}$ on H. Then it is easy to see that $\mathcal{K}^{\mathbb{F}}$ is isomorphic to the category H-Alg of H-algebras and their homomorphisms.

We define the full subcategory H-Alg_{it} of *iterative H-algebras* as follows: an algebra $a : HA \longrightarrow A$ is iterative, if for any $e : X \longrightarrow HX + A$ there is a unique $e^\dagger : X \longrightarrow A$ making the diagram

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[a,A]}"'] \end{tikzcd}$$

commutative. (See [AMV] for the motivation of this concept.)

It can be proved that H-Alg_{it} is reflective in H-Alg. Thus, there exist free iterative H-algebras, RA, on any object A. The monad $\mathbb{R}$ of free iterative algebras, called the *rational monad* of H, is finitary and there is a canonical monad morphism $\alpha : \mathbb{F} \longrightarrow \mathbb{R}$. The category $\mathcal{K}^{\mathbb{R}}$ is identified as the category of *Elgot algebras* in [AMV], i.e., structures $(A, a, (-)^\dagger)$ consisting of a H-algebra (A, a) and a mapping $(-)^\dagger : \mathcal{K}(X, HX + A) \longrightarrow \mathcal{K}(X, A)$ satisfying certain axioms. However, the functor $\alpha_* : \mathcal{K}^{\mathbb{R}} \longrightarrow \mathcal{K}^{\mathbb{F}}$ is not fully faithful in general, although its restriction to $\mathcal{K}_{\mathbb{R}}$ is always fully faithful, see Example 4.3 of [AMV].

Detecting, when the restriction of α_* to Kleisli category is fully faithful, is easy:

Proposition 4.9. *The following are equivalent:*

(1) *The restriction of α_* to $\mathcal{K}_{\mathbb{T}}$ is fully faithful.*
(2) *The functor $K_\alpha : \mathcal{K}_{\mathbb{S}} \longrightarrow \mathcal{K}_{\mathbb{T}}$ is dense.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{K}_{\mathbb{T}} \arrow[r, "K_{\mathbb{T}}"] \arrow[rr, bend left, "Y"] & \mathcal{K}^{\mathbb{T}} \arrow[r, "\widetilde{K_{\mathbb{T}}}"] \arrow[d, "\alpha_*"'] & {[\mathcal{K}_{\mathbb{T}}^{op}, \mathsf{Set}]} \arrow[d, "{[K_\alpha^{op}, \mathsf{Set}]}"] \\ & \mathcal{K}^{\mathbb{S}} \arrow[r, "\widetilde{K_{\mathbb{S}}}"'] & {[\mathcal{K}_{\mathbb{S}}^{op}, \mathsf{Set}]} \end{tikzcd}$$

where the square is the diagram (4.3). The passage from $\mathcal{K}_{\mathbb{T}}$ to $[\mathcal{K}_{\mathbb{S}}^{op}, \mathsf{Set}]$ is the functor $\widetilde{K_\alpha}$ and it is fully faithful if and only if $\alpha_* \cdot K_{\mathbb{T}}$ is fully faithful. ☐

Remark 4.10. Proposition 4.9 and Example 4.8 form a counterexample to Proposition 5.1 of [D]:

Given a dense functor $N : \mathcal{L} \longrightarrow \mathcal{K}_{\mathbb{T}}$ the composite $K_{\mathbb{T}} N : \mathcal{L} \longrightarrow \mathcal{K}^{\mathbb{T}}$ is dense.

Were the above true, then $\alpha_* : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}^{\mathbb{S}}$ would be fully faithful whenever its restriction to $\mathcal{K}_{\mathbb{T}}$ is fully faithful. This is seen as follows: Suppose that the restriction of α_* to $\mathcal{K}_{\mathbb{T}}$ is fully faithful. By Proposition 4.9 we conclude density of K_α. By the above claim the composite $K_{\mathbb{T}} K_\alpha : \mathcal{K}_{\mathbb{S}} \longrightarrow \mathcal{K}^{\mathbb{T}}$ is dense. Thus, the diagonal $\widetilde{K_{\mathbb{T}} K_\alpha}$ of the square

$$\begin{tikzcd} \mathcal{K}^{\mathbb{T}} \arrow[r, "\widetilde{K_{\mathbb{T}}}"] \arrow[d, "\alpha_*"'] & {[\mathcal{K}_{\mathbb{T}}^{op}, \mathsf{Set}]} \arrow[d, "{[K_\alpha^{op}, \mathsf{Set}]}"] \\ \mathcal{K}^{\mathbb{S}} \arrow[r, "\widetilde{K_{\mathbb{S}}}"'] & {[\mathcal{K}_{\mathbb{S}}^{op}, \mathsf{Set}]} \end{tikzcd}$$

is fully faithful. We proved that α_* is fully faithful. By considering $\alpha : \mathbb{F} \longrightarrow \mathbb{R}$ of Example 4.8 we obtain a contradiction.

5. Position of Dense Monad Morphisms in the Category of Monads

In this section we locate dense monad morphisms as those in between the class of strong epimorphisms and the class of epimorphisms in the category

$$\mathsf{Mnd}_{fin}(\mathcal{K})$$

of finitary monads on $\mathcal{K}$ and their morphisms. The main result of this section, Theorem 5.4 below, then characterizes dense monad morphisms by a simple orthogonality condition.

Proposition 5.1. *$\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ is an epimorphism of monads if and only if $\alpha_* : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}^{\mathbb{S}}$ is injective on objects.*

Proof. Form the cokernel of α in $\mathsf{Mnd}_{\mathit{fin}}(\mathcal{K})$ and consider the unique connecting morphism $\tau : \operatorname{coker}(\alpha) \longrightarrow \mathbb{T}$:

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \mathbb{T} \arrow[d, "\mathrm{inr}"] \arrow[rdd, bend left, "\mathrm{id}"] & \\
\mathbb{T} \arrow[r, "\mathrm{inl}"'] \arrow[rrd, bend right, "\mathrm{id}"'] & \operatorname{coker}(\alpha) \arrow[rd, dotted, "\tau"] & \\
& & \mathbb{T}
\end{tikzcd} \tag{5.1}$$

We want to prove that τ is an isomorphism. Since τ is a split epimorphism, it is dense by Lemma 3.9, hence

$$\tau_* : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}^{\operatorname{coker}(\alpha)}$$

is fully faithful.

It therefore suffices to prove that α is an epimorphism if and only if τ_* is bijective on objects. By Proposition 2.9 $\operatorname{coker}(\alpha)$-algebras are pairs $((A,a),(A,b))$ of $\mathbb{T}$-algebras with $\alpha_*(A,a) = \alpha_*(A,b)$. Therefore α_* is injective on objects if and only if $\operatorname{coker}(\alpha)$-algebras are pairs $((A,a),(A,a))$ of $\mathbb{T}$-algebras. Hence α_* is injective on objects if and only if τ_* (or, equivalently, τ) is an isomorphism. $\square$

Corollary 5.2. *Every dense monad morphism is an epimorphism.*

Proof. Suppose $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ is dense and let $\alpha_*(A,a) = \alpha_*(B,b)$. Then necessarily $A = B$ and identity is a morphism from $\alpha_*(A,a)$ to $\alpha_*(A,b)$. Since α_* is fully faithful, identity is a morphism from (A,a) to (A,b). $\square$

Example 5.3. It has been proved by John Isbell [I_1] that the monad morphism $\alpha : \mathbb{S} \longrightarrow \mathbb{T}$, where $\mathbb{S}$ is the monad of semigroups and $\mathbb{T}$ is the monad of monoids, is an epimorphism in $\mathsf{Mnd}_{\mathit{fin}}(\mathsf{Set})$. This can also easily be seen by employing Proposition 5.1, since the forgetful functor

$$\alpha_* : \mathsf{Monoids} \longrightarrow \mathsf{Semigroups}$$

is injective on objects. However, α_* is not fully faithful, thus, the above α is an example of an epimorphism which is not dense in $\mathsf{Mnd}_{\mathit{fin}}(\mathsf{Set})$.

Recall from [K_1] that, in any category, a morphism $a : A \longrightarrow B$ is called *orthogonal* to a morphism $b : C \longrightarrow D$, if for every commutative square

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
C \arrow[r, "b"'] & D
\end{tikzcd}$$

there exists a unique diagonal morphism $d : B \longrightarrow C$ making both triangles commutative.

The orthogonality characterization of dense monad morphisms is the following one:

Theorem 5.4. *The following are equivalent:*

(1) *$\alpha : \mathbb{S} \longrightarrow \mathbb{T}$ is dense.*
(2) *α is orthogonal to every monad morphism of the form $\langle \pi_A, \pi_B \rangle : \{\{f,f\}\} \longrightarrow \langle\!\langle A,A \rangle\!\rangle \times \langle\!\langle B,B \rangle\!\rangle$.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\alpha"] \arrow[d] & \mathbb{T} \arrow[d, "\langle \check{a}, \check{b} \rangle"] \\
\{\{f,f\}\} \arrow[r, "\langle \pi_A, \pi_B \rangle"'] & \langle\!\langle A,A \rangle\!\rangle \times \langle\!\langle B,B \rangle\!\rangle
\end{tikzcd}$$

in $\mathsf{Mnd}_{\mathit{fin}}(\mathcal{K})$. Then the (necessarily unique) diagonal monad morphism $\mathbb{T} \longrightarrow \{\{f,f\}\}$ witnesses precisely the fact that α_* is fully faithful. $\square$

Recall from [K_1] that strong epimorphisms are defined as those morphisms that are orthogonal to every monomorphism. The above theorem has then the following corollary:

Corollary 5.5. *Every strong epimorphism in* $\mathsf{Mnd}_{fin}(\mathcal{K})$ *is dense.*

Proof. Observe that every morphism of the form $\langle \pi_A, \pi_B \rangle : \{\{f, f\}\} \longrightarrow \langle\!\langle A, A \rangle\!\rangle \times \langle\!\langle B, B \rangle\!\rangle$ is a monomorphism. $\square$

Example 5.6. We provide an example of a dense monad morphism that is not strong epimorphic.

We will use the fact that the category $\mathsf{Fin}(\mathsf{Set}, \mathsf{Set})$ is equivalent to the category $[\mathsf{Set}_{fp}, \mathsf{Set}]$, where $E : \mathsf{Set}_{fp} \longrightarrow \mathsf{Set}$ is the inclusion of a small full subcategory representing finite sets.

First observe that regular epimorphisms in the category $[\mathsf{Set}_{fp}, \mathsf{Set}]$ coincide with (pointwise) epimorphisms. Thus, the horizontal composite $\tau\sigma = \tau K' \cdot H\sigma = H'\sigma \cdot \tau K : HK \longrightarrow H'K'$ is a regular epimorphism for every pair $\tau : H \longrightarrow H'$, $\sigma : K \longrightarrow K'$ of regular epimorphisms.

Observe further that the functor $H \mapsto F_H$ on $\mathsf{Fin}(\mathsf{Set}, \mathsf{Set})$ which assigns to every finitary endofunctor H the underlying functor F_H of the free monad $\mathbb{F}_H$ on H preserves regular epimorphisms: for let $\tau : H \longrightarrow H'$ be pointwise an epimorphism. Then we have a chain τ_0, τ_1, ..., of connecting epimorphisms:

$$\begin{tikzcd}
W_0 \arrow[r, "w_{0,1}"] \arrow[d, "\tau_0"'] & W_1 \arrow[r, "w_{1,2}"] \arrow[d, "\tau_1"'] & W_2 \arrow[r, "w_{2,3}"] \arrow[d, "\tau_2"'] & {} & \cdots & {} \arrow[r, "w_{k-1,k}"] & W_k \arrow[r, "w_{k,k+1}"] \arrow[d, "\tau_k"'] & W_{k+1} \arrow[r, "w_{k+1,k+2}"] \arrow[d, "\tau_{k+1}"'] & {} & \cdots \\
W'_0 \arrow[r, "w'_{0,1}"'] & W'_1 \arrow[r, "w'_{1,2}"'] & W'_2 \arrow[r, "w'_{2,3}"'] & {} & \cdots & {} \arrow[r, "w'k-1{,}k"'] & W'_k \arrow[r, "w'_{k,k+1}"'] & W'_{k+1} \arrow[r, "w'_{k+1,k+2}"'] & {} & \cdots
\end{tikzcd}$$

where the horizontal chains have F_H, resp. $F_{H'}$ as a colimit, i.e., $W_0 = W'_0 = \mathrm{Id}$, $W_{k+1} = HW_k + \mathrm{Id}$, $W'_{k+1} = HW'_k + \mathrm{Id}$, and $w_{0,1} = \mathrm{inr} : \mathrm{Id} \longrightarrow H + \mathrm{Id}$, $w'_{0,1} = \mathrm{inr} : \mathrm{Id} \longrightarrow H' + \mathrm{Id}$, $w_{k+1,k+2} = Hw_{k,k+1} + \mathrm{id}$. The connecting morphisms are defined by putting $\tau_0 = \mathrm{id} : \mathrm{Id} \longrightarrow \mathrm{Id}$ and $\tau_{k+1} = \tau\tau_k + \mathrm{id} : W_{k+1} \longrightarrow W'_{k+1}$ and they are all pointwise epimorphisms. So is the induced connecting morphism $F_\tau : \mathrm{colim}\, W_k \longrightarrow \mathrm{colim}\, W'_k$.

Thus, a monad morphism is a regular epimorphism in $\mathsf{Mnd}_{fin}(\mathsf{Set})$ if and only if it is a regular epimorphism as a natural transformation in $\mathsf{Fin}(\mathsf{Set}, \mathsf{Set})$, see. e.g., Theorem 21.6.3(d) of [S].

Therefore regular epimorphisms in $\mathsf{Mnd}_{fin}(\mathsf{Set})$ are closed under composition and thus strong epimorphisms coincide with regular epimorphisms in $\mathsf{Mnd}_{fin}(\mathsf{Set})$ by Proposition 3.8 of [K_1].

We conclude that the full inclusion $\alpha_* : \mathsf{Groups} \longrightarrow \mathsf{Monoids}$ is an example of a dense α such that α is *not* strong epimorphic. This follows from the fact that α is not (regular) epi in $\mathsf{Fin}(\mathsf{Set}, \mathsf{Set})$.

6. Possible Generalizations

The above results can be easily generalized in two ways:

(1) All main results of this paper go through verbatim if one systematically replaces locally finitely presentable categories and finitary monads by *locally* $\mathbb{D}$*-presentable categories* and $\mathbb{D}$*-accessible monads* for a limit doctrine $\mathbb{D}$, as introduced in [ABLR].

Besides an obvious generalization to λ*-presentable categories* and λ*-accessible monads* for an uncountable regular cardinal λ (see [GU]) we obtain thus, e.g., results for the interesting doctrine $\mathbb{D}$ of finite products (replacing thus filtered colimits with *sifted colimits*).

(2) One can also start with a symmetric monoidal closed category $\mathcal{V}$ that is locally finitely presentable as a monoidal category (see [K_3]) and choose it as a base category for category theory *enriched over* $\mathcal{V}$, see [K_2]. An example of such a base category is the category Ab of Abelian groups and group homomorphisms. An Ab-category is then one that has Abelian groups as hom-objects and its composition maps are group homomorphisms.

Replacing systematically categories and functors by $\mathcal{V}$-categories and $\mathcal{V}$-functors one obtains the corresponding results on dense morphisms between $\mathcal{V}$-monads.

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