Complete monotonicity and related properties of some special functions

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Outline

1 Motivation
   - Estimation of trigonometric sums of hypergeometric type
   - Completely monotonic functions

2 The special functions $\Delta_s, t(x)$ and $L_s, t(x)$
   - Main results
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   - Main results
   - Sketch of the proofs
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Motivation

The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Estimation of trigonometric sums of hypergeometric type

Completely monotonic functions

Motivation

Problem

Let $s, t > 0$. For $m, n \in \mathbb{N}$ with $m > n \geq 2$, find estimates for

$$U_{n,m}(x) := \sum_{k=n}^{m} \frac{(t)^k}{(s)_k} e^{ikx},$$

using estimates for

$$V_{n,m}(x) := \frac{\Gamma(s)}{\Gamma(t)} \sum_{k=n}^{m} \frac{1}{k^{s-t}} e^{ikx},$$

where $(a)_k = a(a + 1) \ldots (a + k - 1) = \frac{\Gamma(a + k)}{\Gamma(a)}$ is the Pochhammer symbol.
For $s, t > 0$ and $k \in \mathbb{N}$ we define

$$\Delta_{s, t}(k) = \frac{\Gamma(s)}{\Gamma(t)} \frac{1}{k^{s-t}} - \frac{(t)_k}{(s)_k} = \frac{\Gamma(s)}{\Gamma(t)} \left[ \frac{1}{k^{s-t}} - \frac{\Gamma(k + t)}{\Gamma(k + s)} \right]$$

For appropriate $s, t > 0$, it is possible to obtain the sharp estimate

$$\left| \sum_{k=n}^{m} \Delta_{s, t}(k) e^{ikx} \right| \leq \frac{1}{\sin \frac{x}{2}} \frac{1}{n^{s-t+1}} \frac{\Gamma(s)}{\Gamma(t)} \frac{(s - t)(s + t - 1)}{2}$$

for all $x \in (0, 2\pi)$ and $m, n \in \mathbb{N}$ with $m > n > 1$.

using complete monotonicity of the sequence $\Delta_{s, t}(k)$. 
Completely monotonic functions

- A function $f : (0, \infty) \to \mathbb{R}$ is called *completely monotonic* if $f$ has derivatives of all orders and satisfies
  
  $$(-1)^n f^{(n)}(x) \geq 0, \text{ for all } x > 0 \text{ and } n = 0, 1, 2, \ldots$$

- A characterization of completely monotonic functions:

**Theorem (Bernstein)**

$f$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} \, d\mu(t),$$

where $\mu$ is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. 

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Motivation

The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

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Motivation

The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Estimation of trigonometric sums of hypergeometric type

Completely monotonic functions

Completely monotonic functions of positive order

Definition (S.K and H. L. Pedersen, 2009)

Let $\alpha \geq 0$. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called a completely monotonic function of order $\alpha$ if $x^\alpha f(x)$ is completely monotonic on $(0, \infty)$.

We recall also that the Riemann-Liouville fractional integral $I_\alpha(\mu)(t)$ of order $\alpha > 0$, of a Borel measure $\mu$ on $[0, \infty)$ is defined by

$$I_\alpha(\mu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\mu(s).$$
Completely monotonic functions of positive order

**Definition (S.K and H. L. Pedersen, 2009)**

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Theorem (S.K and H. L. Pedersen, 2009)

The function $f : (0, \infty) \to \mathbb{R}$ is completely monotonic of order $\alpha > 0$ if and only if $f$ is the Laplace transform of a fractional integral of order $\alpha$ of a positive Radon measure $\mu$ on $[0, \infty)$, that is,

$$f(x) = \int_0^\infty e^{-xt} l_\alpha(\mu)(t) \, dt.$$
Corollary

Let $r$ be an integer $\geq 2$. The function $f(x)$ is completely monotonic of order $r$ on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} p(t) \, dt,$$

where the integral converges for all $x > 0$ and $p(t)$ is $r - 2$ times continuously differentiable on $[0, \infty)$ with

$$p^{(r-2)}(t) = \int_0^t \mu([0, s]) \, ds$$

for some Radon measure $\mu$ and $p^{(k)}(0) = 0$ for $0 \leq k \leq r - 2$. 
The function $x f(x)$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} p(t) \, dt,$$

where $p(t)$ is nonnegative and increasing on $[0, \infty)$ and the integral converges for all $x > 0$.

This is equivalent to the fact that

$$(-1)^n x^{n+1} f^{(n)}(x)$$

is nonnegative and decreasing on $(0, \infty)$ for all $n = 0, 1, 2, \ldots$. 
The function $xf(x)$ is completely monotonic on $(0, \infty)$ if and only if

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This is equivalent to the fact that

$$(-1)^n x^{n+1} f^{(n)}(x)$$

is nonnegative and decreasing on $(0, \infty)$ for all $n = 0, 1, 2, \ldots$. 
The function $x^2 f(x)$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} p(t) \, dt,$$

where $p(t)$ is nonnegative, increasing and convex on $[0, \infty)$ with $p(0) = 0$ and the integral converges for all $x > 0$. 
Bernstein functions

Definition
A function \( f : (0, \infty) \rightarrow (0, \infty) \) is called a Bernstein function if \( f \) has derivatives of all orders and \( f' \) is completely monotonic on \((0, \infty)\).

Example
\( L_t(x) = \frac{xt}{x+t} \) is a Bernstein function on \((0, \infty)\) for all \( t > 0 \).
Bernstein functions

Definition
A function $f : (0, \infty) \rightarrow (0, \infty)$ is called a Bernstein function if $f$ has derivatives of all orders and $f'$ is completely monotonic on $(0, \infty)$.

Example
$L_t(x) = \frac{xt}{x + t}$ is a Bernstein function on $(0, \infty)$ for all $t > 0$. 
The special functions $\Delta_s$, $t(x)$ and $L_s$, $t(x)$

Characterization of Bernstein functions

**Theorem**

$f : (0, \infty) \rightarrow (0, \infty)$ is a Bernstein function iff

$$f(x) = ax + b + \int_0^\infty (1 - e^{-xt}) d\nu(t),$$

where $a, b$ are nonnegative constants and $\nu$, called the Lévy measure, is a positive measure on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{t}{1 + t} d\nu(t) < \infty.$$
The functions $\Delta_{s,t}(x)$ and $L_{s,t}(x)$

For $s, t > 0$ and $x > 0$ we set

$$\Delta_{s,t}(x) := \frac{\Gamma(s)}{\Gamma(t)} \left[ \frac{1}{x^{s-t}} - \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]$$

For all $x > 0$ we have

$$\Delta_{s,t}(x) = \frac{\Gamma(s)}{\Gamma(t)} \frac{1}{x^{s-t+1}} L_{s,t}(x),$$

where

$$L_{s,t}(x) = x - \frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t+1}.$$
The asymptotic behaviour of $\Delta_s, t(x)$ and $L_s, t(x)$ as $x \rightarrow \infty$

We have

$$\lim_{x \to \infty} L_s, t(x) = \frac{(s - t)(s + t - 1)}{2}.$$

$$\Delta_s, t(x) = O\left(\frac{1}{x^{s-t+1}}\right), \text{ as } x \rightarrow \infty.$$
Let $E := \{(s, t) \in \mathbb{R}^2 : s, t > 0\}$.

1. Determine the set of $(s, t) \in E$ so that
$$0 < L_{s, t}(x) < \frac{(s - t)(s + t - 1)}{2}, \text{ holds for all } x > 0.$$ 

2. Determine the set of $(s, t) \in E$ so that $L_{s, t}(x)$ is strictly increasing and concave on $(0, \infty)$.

3. Determine the set of $(s, t) \in E$ so that $L_{s, t}(x)$ is a Bernstein function on $(0, \infty)$. 
Let \( E := \{(s, t) \in \mathbb{R}^2 : s, t > 0 \} \).

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3. Determine the set of \((s, t) \in E\) so that \(L_{s,t}(x)\) is a Bernstein function on \((0, \infty)\).
Some important sets

**Definition**

\[ P := \{(s, t) \in E : s + t \geq 1\}, \quad H := \{(s, t) \in P : s > t\}, \]
\[ P_1 := \{(s, t) \in E : s - t > 1\}, \]
\[ P_2 := \{(s, t) \in P : 0 < s - t < 1\}. \]

Moreover, if \( M \) is any subset of \( \mathbb{R}^2 \), then we will denote by \( M^* \) its reflection with respect to the line \( \{(s, t) \in \mathbb{R}^2 : s = t\} \).
Figure: \( H \) is equal to the union of \( P_1 \) and \( P_2 \). \( H^* \) is equal to the union of \( P_1^* \) and \( P_2^* \). \( P \) is equal to the union of \( P_1, P_2, P_1^*, \) and \( P_2^* \).
The special functions $\Delta_{s,t}(x)$ and $L_{s,t}(x)$

Complete monotonicity of $\Delta_{s,t}(x)$

Theorem (S. K., 2008)

Let $(s, t) \in E$ with $t \neq s$.
1. The function $\Delta_{s,t}(x)$ is completely monotonic on $(0, \infty)$ if and only if $(s, t) \in H$.
2. The function $\Delta_{s,t}(x)$ changes sign in $(0, \infty)$ precisely when $(s, t) \in E \setminus P$.
3. We have $\Delta_{s,t}(x) < 0$ for all $x > 0$ if and only if $(s, t) \in H^*$.

Corollary

Let $(s, t) \in E$ with $t \neq s$. We have $L_{s,t}(x) > 0$ for all $x > 0$ if and only if $(s, t) \in H$, whereas $L_{s,t}(x) < 0$ for all $x > 0$ if and only if $(s, t) \in H^*$. The function $L_{s,t}(x)$ changes sign in $(0, \infty)$ precisely when $(s, t) \in E \setminus P$. 

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Complete monotonicity of $\Delta_s, t(x)$

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2. The function $\Delta_s, t(x)$ changes sign in $(0, \infty)$ precisely when $(s, t) \in E \setminus P$.

3. We have $\Delta_s, t(x) < 0$ for all $x > 0$ if and only if $(s, t) \in H^*$.

**Corollary**

Let $(s, t) \in E$ with $t \neq s$. We have $L_s, t(x) > 0$ for all $x > 0$ if and only if $(s, t) \in H$, whereas $L_s, t(x) < 0$ for all $x > 0$ if and only if $(s, t) \in H^*$. The function $L_s, t(x)$ changes sign in $(0, \infty)$ precisely when $(s, t) \in E \setminus P$. 
A well-known asymptotic formula

\[
\frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t} = \\
1 - \frac{(s - t)(s + t - 1)}{2x} + \frac{(s - t)(s - t + 1) p(s, t)}{24x^2} + O\left(\frac{1}{x^3}\right),
\]

as \( x \to \infty \), where

\[
p(s, t) := 3s^2 + 6st + 3t^2 - 5s - 7t + 2
\]
The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Main results

Sketch of the proofs

The parabola $p(s, t) = 0$ and the critical lines
Definition

Let

\[ A := \{(s, t) \in P : p(s, t) \geq 0\} \]

and set

\[ A_1 := H \cap A, \quad A_2 := P_2^* \cap A, \quad A_3 := P_1^* \cap A \]
The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Main results
Sketch of the proofs

Figure: $A$ is equal to the union of $A_1$, $A_2$, and $A_3$. The dotted curves describe the parts of the parabola $p(s, t) = 0$ that lie outside $P$. 

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Main result 1

Theorem (S. K. and M. Lamprecht, 2010)

- For \((s, t) \in P_1\) the function \(L_{s, t}(x)\) is strictly increasing and concave on \((0, \infty)\) but not a Bernstein function. In this case we have
  \[
  0 < L_{s, t}(x) < \frac{(s - t)(s + t - 1)}{2} \quad \text{for all} \quad x > 0. \tag{1}
  \]

- For \((s, t) \in A_1 \setminus P_1\) the function \(L_{s, t}(x)\) is a Bernstein function on \((0, \infty)\) and
  \[
  0 < L_{s, t}(x) < \frac{(s - t)(s + t - 1)}{2} \quad \text{for all} \quad x > 0. \tag{2}
  \]
Main result 1, continued

**Theorem**

- For \((s, t) \in A_2\) the function \(-L_{s, t}(x)\) is a Bernstein function on \((0, \infty)\) and
  \[
  \frac{(s - t)(s + t - 1)}{2} < L_{s, t}(x) < 0 \quad \text{for all } x > 0. \tag{3}
  \]

- For \((s, t) \in A_3\) the function \(-L_{s, t}(x)\) is completely monotonic on \((0, \infty)\) and
  \[
  -\infty < L_{s, t}(x) < \frac{(s - t)(s + t - 1)}{2} \quad \text{for all } x > 0. \tag{4}
  \]
The sharpness of the result

Remark

All bounds in the inequalities (1)–(4) are sharp.

For \((s, t) \in E \setminus A\) with \(t \neq s, s + 1\), the derivatives \(L'_{s, t}(x)\) and \(L''_{s, t}(x)\) change sign in \((0, \infty)\) and inequalities (1)–(4) fail to hold for appropriate \(x > 0\).
Monotonicity of $L_{s,t}(x)$

\[ L'_{s,t}(x) = 1 - \frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t} \left[ x \left( \psi(x + t) - \psi(x + s) \right) + s - t + 1 \right], \]

\[ L''_{s,t}(x) = -\frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t+1} \Phi_{s,t}(x), \]

\[ \Phi_{s,t}(x) : = \left( \psi(x + t) - \psi(x + s) + \frac{s - t + 1}{x} \right)^2 + \left( \psi(x + t) - \psi(x + s) + \frac{s - t + 1}{x} \right)', \]

where $\psi(x) = \Gamma'(x)/\Gamma(x)$

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Motivation

The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Main results

Sketch of the proofs

Monotonicity of $L_s, t(x)$

$L'_s, t(x) = 1 - \frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t} \left[ x \left( \psi(x + t) - \psi(x + s) \right) + s - t + 1 \right],$

$L''_s, t(x) = -\frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t+1} \Phi_s, t(x),$

$\Phi_s, t(x) : = \left( \psi(x + t) - \psi(x + s) + \frac{s - t + 1}{x} \right)^2$

$+ \left( \psi(x + t) - \psi(x + s) + \frac{s - t + 1}{x} \right)'$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$
Main result 2
Complete monotonicity of $\Phi_{s, t}(x)$ of positive order

Theorem (S. K. and M. Lamprecht, 2010)

1. For $(s, t) \in A_1$ the function $\Phi_{s, t}(x)$ is completely monotonic of order 2 on $(0, \infty)$.

2. For $(s, t) \in A_2$ the function $-\Phi_{s, t}(x)$ is completely monotonic of order 1 on $(0, \infty)$.

3. For $(s, t) \in A_3$ the function $\Phi_{s, t}(x)$ is completely monotonic of order 1 on $(0, \infty)$.

4. For $(s, t) \in E \setminus A$ with $t \neq s, s + 1$, the function $\Phi_{s, t}(x)$ changes sign in $(0, \infty)$.
Motivation

The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Main results

Sketch of the proofs

Figure: $A$ is equal to the union of $A_1$, $A_2$, and $A_3$. The dotted curves describe the parts of the parabola $p(s, t) = 0$ that lie outside $P$. 

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The special functions $\Delta_s$, $t(x)$ and $L_s$, $t(x)$

**Motivation**

**Main results**

**Sketch of the proofs**

\[
\Phi_{s, t}(x) = \int_0^\infty e^{-xu} F_{s, t}(u) \, du,
\]

where

\[
F_{s, t}(u) := \int_0^u \sigma_{s, t}(u - v) \sigma_{s, t}(v) \, dv - u \sigma_{s, t}(u),
\]

\[
F'_{s, t}(u) = \int_0^u \sigma'_{s, t}(u - v) \sigma_{s, t}(v) \, dv - u \sigma'_{s, t}(u),
\]

\[
F''_{s, t}(u) = u \varphi''_{s, t}(u) + \int_0^u \varphi'_{s, t}(u - v) \varphi'_{s, t}(v) \, dv,
\]

with

\[
\varphi_{s, t}(u) := \frac{e^{(1-t)u} - e^{(1-s)u}}{e^u - 1}, \quad \varphi_{s, t}(0) = s - t,
\]

\[
\sigma_{s, t}(u) := \varphi_{s, t}(0) - \varphi_{s, t}(u) + 1.
\]
\[ \Phi_{s,t}(x) = \int_{0}^{\infty} e^{-xu} F_{s,t}(u) \, du, \]

where

\[
F_{s,t}(u) := \int_{0}^{u} \sigma_{s,t}(u - v) \sigma_{s,t}(v) \, dv - u \sigma_{s,t}(u),
\]

\[
F'_{s,t}(u) = \int_{0}^{u} \sigma'_{s,t}(u - v) \sigma_{s,t}(v) \, dv - u \sigma'_{s,t}(u),
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F''_{s,t}(u) = u \varphi''_{s,t}(u) + \int_{0}^{u} \varphi'_{s,t}(u - v) \varphi'_{s,t}(v) \, dv, \text{ with}
\]

\[
\varphi_{s,t}(u) := \frac{e^{(1-t)u} - e^{(1-s)u}}{e^u - 1}, \quad \varphi_{s,t}(0) = s - t,
\]

\[
\sigma_{s,t}(u) := \varphi_{s,t}(0) - \varphi_{s,t}(u) + 1.
\]
The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Main results
Sketch of the proofs

The fundamental Lemma

Lemma

1. For $(s, t) \in A_1$ we have $F''_{s, t}(u) > 0$ for all $u > 0$.
2. For $(s, t) \in A_2$ we have $F'_{s, t}(u) < 0$ for all $u > 0$.
3. For $(s, t) \in A_3$ we have $F'_{s, t}(u) > 0$ for all $u > 0$.
Monotonicity of $\varphi_s, t(u)$

Lemma

1. For $(s, t) \in H$ the function $\varphi_s, t(u)$ is strictly decreasing on $(0, \infty)$.

2. For $(s, t) \in H^*$ the function $\varphi_s, t(u)$ is strictly increasing on $(0, \infty)$.

3. For $(s, t) \in E \setminus P$ with $t \neq s$ the function $\varphi'_s, t(u)$ changes sign in $(0, \infty)$. 

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Figure: $H$ is equal to the union of $P_1$ and $P_2$. $H^*$ is equal to the union of $P_1^*$ and $P_2^*$. $P$ is equal to the union of $P_1$, $P_2$, $P_1^*$, and $P_2^*$. 
Convexity of \( \varphi_{s, t}(u) \)

**Lemma**

Let \( \varepsilon(s, t) := 2s^2 + 2st + 2t^2 - 3s - 3t + 1 \) and

\[
K := H \cap \{(s, t) \in \mathbb{R}^2 : \varepsilon(s, t) \geq 0\}.
\]

1. For \( (s, t) \in K \) we have \( \varphi''_{s, t}(u) > 0 \) for all \( u > 0 \).
2. For \( (s, t) \in K^* \) we have \( \varphi''_{s, t}(u) < 0 \) for all \( u > 0 \).
3. For \( (s, t) \in E \setminus (K \cup K^*) \) with \( t \neq s \) the function \( \varphi''_{s, t}(u) \) changes sign in \((0, \infty)\).

Note that \( \varphi''_{s, t}(0) = \frac{1}{6} (s - t) \varepsilon(s, t) \).
Figure: The sets $K$ and $K^*$. The dotted curve describes the part of the ellipse $\varepsilon(s, t) = 0$ that lies outside $P$. The hatched area is $A_1 \setminus K$. 
The parabola $p(s, t) = 0$ and the ellipse $\epsilon(s, t) = 0$
The function $\Xi_{s,t}(u)$

$\Xi_{s,t}(u) := \sigma'_{s,t}(u)\sigma'''_{s,t}(u) - (\sigma''_{s,t}(u))^2, \quad s, t, u \in \mathbb{R}.$

Lemma

For $(s, t) \in P_2 \cup P_2^*$ we have $\Xi_{s,t}(u) < 0$ for all $u \in (0, \infty)$.

For $(s, t) \in A_3$ we have $\Xi_{s,t}(u) > 0$ for all $u \in (0, \infty)$. 
The special functions $\Delta_{s,t}(x)$ and $L_{s,t}(x)$

Main results

Sketch of the proofs

The function $\Theta_{s,t}(u)$

**Lemma**

For $s, t, u \in \mathbb{R}$ let

$$\Theta_{s,t}(u) := \sigma_{s,t}(u)\sigma''_{s,t}(u) - (\sigma'_{s,t}(u))^2.$$  

If $(s, t) \in A_2$, then $\Theta_{s,t}(u) > 0$ for all $u > 0$.

If $(s, t) \in A_3$, then $\Theta_{s,t}(u) < 0$ for all $u > 0$.

If $(s, t) \in H^* \setminus (\overline{A_2 \cup A_3})$, then $\Theta_{s,t}(u)$ changes sign in $(0, \infty)$.  

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Motivation

The special functions $\Delta_s, t(x)$ and $L_s, t(x)$

Main results

Sketch of the proofs

Figure: $A$ is equal to the union of $A_1$, $A_2$, and $A_3$. The dotted curves describe the parts of the parabola $p(s, t) = 0$ that lie outside $P$. 

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Sketch of the proof of Main result 2

Recall that

$$\Phi_{s,t}(x) = \int_{0}^{\infty} e^{-xu} F_{s,t}(u) \, du.$$ 

For \((s, t) \in A_1\) we have

$$F''_{s,t}(u) > 0, \quad F'_{s,t}(u) > 0, \quad F_{s,t}(u) > 0 = F_{s,t}(0), \quad \text{for all } u > 0.$$

Therefore \(x^2 \Phi_{s,t}(x)\) is completely monotonic on \((0, \infty)\)

In a similar manner, we handle the cases \((s, t) \in A_2\) and \((s, t) \in A_3\).
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The asymptotic behaviour of $\Phi_{s, t}(x)$ for $x \to 0^+$ and $x \to \infty$.

We have

1. $$\lim_{x \to 0^+} x^2 \Phi_{s, t}(x) = (s - t)(s - t + 1).$$

2. $$\lim_{x \to \infty} x^4 \Phi_{s, t}(x) = \frac{1}{12} (s - t)(s - t + 1) p(s, t).$$
Recall that \( L_{s, t}(x) = x - \frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t+1} \).

\[
\lim_{x \to \infty} x^2 L'_{s, t}(x) = \frac{1}{24} (s-t)(s-t+1) p(s, t), \quad \text{for all } s, t > 0,
\]

\[
\lim_{x \to 0} L'_{s, t}(x) = \begin{cases} 
1, & s > t, \\
-\infty, & 0 < s - t + 1 < 1, \\
+\infty, & s - t + 1 < 0,
\end{cases}
\]
**Motivation**

The special functions $\Delta_{s,t}(x)$ and $L_{s,t}(x)$

**Main results**

Sketch of the proofs

\[ \lim_{x \to \infty} -x^3 L''_{s,t}(x) = \frac{1}{12} (s-t) (s-t+1) p(s,t), \text{ for all } s, t > 0 \]

\[ \lim_{x \to 0} -L''_{s,t}(x) = \begin{cases} 
0, & s - t > 1 \\
+\infty, & 0 < s - t < 1, \\
-\infty, & 0 < s - t + 1 < 1, \\
+\infty, & s - t + 1 < 0. 
\end{cases} \]
Sketch of the proof of Main result 1

- For \((s, t) \in A_1\) we write

\[
-L''_{s, t}(x) = \frac{\Gamma(x + t)}{\Gamma(x + s)} \frac{1}{x^{1-s+t}} x^2 \Phi_{s, t}(x)
\]

- For \((s, t) \in A_2 \cup A_3\) we write

\[
L''_{s, t}(x) = -\frac{\Gamma(x + t)}{\Gamma(x + s)} x^{s-t} x \Phi_{s, t}(x).
\]
Motivation
The special functions $\Delta_s, \, t(x)$ and $L_s, \, t(x)$

Main results
Sketch of the proofs

The end

Thank you very much for your attention
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