Nonlinear dynamic analysis of heterogeneous orthotropic membranes by the analog equation method

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Abstract

In this paper the Analog Equation Method (AEM), a BEM-based method, is employed to analyze the dynamic response of flat heterogeneous orthotropic membranes of arbitrary shape, undergoing large deflections. The problem is formulated in terms of the three displacement components. Due to the heterogeneity of the membrane, the elastic constants are position dependent and consequently the coefficients of the partial differential equations governing the dynamic equilibrium of the membrane are variable. Using the concept of the analog equation, the three-coupled nonlinear second order hyperbolic partial differential equations are replaced with three uncoupled Poisson's quasi-static equations with fictitious time dependent sources. The fictitious sources are represented by radial base functions series and are established using a BEM-based procedure. Both free and forced vibrations are considered. Membranes of various shapes are analyzed to illustrate the merits of the method as well as its applicability, efficiency and accuracy. The proposed method is boundary-only in the sense that the discretization and the integration are restricted on the boundary. Therefore, it maintains all the advantages of the pure BEM.

1 Introduction

The modern technology has provided us with light materials that can withstand great tensile forces. This enables the use of tensile structures to cover spans with dimensions of about 100 m. Among them the membranes play an important role.
Therefore a rigorous analysis to predict their response under static and dynamic loads becomes very important.

In the linear membrane theory we assume that the additional stretching of the membrane due to the in-service transverse load is small and the stress resultants are predetermined and remain unchanged during the out-of-plane deformation. However, with increasing the transverse load, the additional stretching of the membrane cannot be neglected. A consequence of this is the coupling of the transverse displacement with the in-plane ones and the govern differential equations are coupled and nonlinear. Modern materials, mostly composite, exhibit anisotropy. Heterogeneity may also appear, which may arise either from the material or from the variable thickness of the membrane. In the later case the equations are more complicated, since the heterogeneity is reflected in the coefficients of the differential equations, which become position dependent.

The solution of the membrane equations is a very difficult mathematical problem. Analytical solutions as well as approximate ones for the static problem of isotropic and homogeneous membrane are listed in [1]. Numerical solutions for the same problem have been obtained by the FEM [2] and by the AEM [1]. Moreover, AEM has been employed for the static problem of orthotropic and heterogeneous membranes [3].

The membranes under static loads preclude instability. However, dynamic loads complicate the analysis. The effect of the airflow has great practical significance, because negative loads produced by the wind (suction) may reach up to 1.1 \( kN/m^2 \). Under loads of such magnitude membranes often experience extremely strong vibrations. Vortex shedding may produce harmonic lift forces. Flutter and galloping are wind effects that require also dynamic analysis. As the membranes undergo large deflections, the linear dynamic analysis is inadequate to predict their response and to study their dynamic characteristics. Hence, nonlinear vibration analysis is necessary. The governing equations are coupled nonlinear partial differential equations of hyperbolic type. Therefore their solution raises a much more difficult problem compared to the static one. With regard to the dynamic problem there are FEM formulations [2] but, to the authors knowledge, no numerical results have been published. The AEM has been successfully employed to solve the dynamic problem of isotropic membranes, linear heterogeneous [4] as well as of nonlinear homogeneous [5].

In this paper the AEM is employed to nonlinear dynamic analysis of heterogeneous orthotropic membranes. According to this method, the three-coupled nonlinear partial differential equations of hyperbolic type are replaced with three equivalent quasi-static Poisson's equations under fictitious time dependent loads. These fictitious loads are represented by radial base function series and are established using the procedure presented in [5]. Subsequently, the displacements and the stress resultants at any point and instant are computed from their integral representations, which are used as mathematical formulas. Several membranes are analyzed to illustrate the merits of the method and its capabilities. The developed method is boundary only in the sense that the discretization and integration are limited only to the boundary maintaining, thus, all the advantage of the pure BEM.
2 Problem statement and governing equations

Consider a thin flexible initially flat elastic membrane consisting of heterogeneous orthotropic linearly elastic material having surface mass density \( \rho(x,y) \) occupying the two-dimensional, in general multiply connected, domain \( \Omega \) in the \( xy \)-plane bounded by the \( K + 1 \) nonintersecting contours \( \Gamma_0, \Gamma_1, \ldots, \Gamma_K \). The membrane is prestressed either by imposed displacement \( \hat{u}, \hat{v} \) or by external forces \( \vec{T}_x, \vec{T}_y \) acting along the boundary \( \Gamma = \bigcup_{j=0}^K \Gamma_j \). Moderate large deflections are considered. They result from nonlinear kinematic relations, which retain the square of the slopes of the deflection surface, while the strain components remain still small compared with the unity. This theory is good for considerably large deflections.

\[
\begin{align*}
\varepsilon_x &= u_x + \frac{1}{2} w_{xx}, & \varepsilon_y &= v_y + \frac{1}{2} w_{yy}, & \gamma_{xy} &= u_y + v_x + w_{xy} \quad (1a,b,c)
\end{align*}
\]

where \( u = u(x,y), \ v = v(x,y) \) are the in-plane displacement component and \( w = w(x,y,t) \) the transverse deflection due to load \( g = g(x,y,t) \) acting in the direction normal to its plane.

Using Hamilton’s principle and neglecting the in-plane inertia forces we obtain the following differential equations, which govern the dynamic equilibrium of the membrane

\[
\begin{align*}
N_{xx} + N_{yy} &= 0 \quad (2a) \\
N_{yx} + N_{xy} &= 0 \quad (2b)
\end{align*}
\]

\[
\rho w_{tt} - N_x w_{xx} - 2N_{xy} w_{xy} - N_y w_{yy} = g \quad (2c)
\]

in \( \Omega \), together with the boundary conditions

\[
\begin{align*}
T_x &= \vec{T}_x \quad \text{or} \quad u = \tilde{u} \quad (3a) \\
T_y &= \vec{T}_y \quad \text{or} \quad v = \tilde{v} \quad (3b)
\end{align*}
\]

\[
T_x w_{xx} + T_y w_{yy} = \vec{V} \quad \text{or} \quad w = \tilde{w} \quad (3c)
\]

on \( \Gamma \), and the initial conditions

\[
w(x,y,0) = \tilde{w}_0, \quad \dot{w}(x,y,0) = \ddot{w}_0 \quad \text{in} \ \Omega \quad (4a,b)
\]

The quantities

\[
N_x = C_1 \varepsilon_x + C_x, \quad N_y = C_1 \varepsilon_y + C_x, \quad N_{xy} = C_{12} \gamma_{xy} \quad (5a,b,c)
\]

are the membrane forces, in which

\[
C_1 = \frac{E_h}{1 - \nu_1 \nu_2}, \quad C_2 = \frac{E_2 h}{1 - \nu_1 \nu_2}, \quad C = \frac{E_1 \nu_2 h}{1 - \nu_1 \nu_2}, \quad C_{12} = \frac{E_2 \nu_1 h}{1 - \nu_1 \nu_2}, \quad C_{12} = G h \quad (6a,b,c,d)
\]
are the stiffness coefficients of the orthotropic membrane with $E_1, E_2$ and $v_1, v_2$ are the elastic moduli and the Poisson coefficients in the $x$ and $y$ directions, respectively, constraint by the relation $E_1v_1 = E_2v_2$, and $G$ is the shear modulus [6]. Note that for heterogeneous material the stiffness coefficients are position dependent, namely $C_1 = C_1(x, y)$, $C_2 = C_2(x, y)$, $C = C(x, y)$ and $C_{12} = C_{12}(x, y)$. Moreover,

$$T_x = N_x \cos a + N_{xy} \sin a, \quad T_y = N_{xy} \cos a + N_y \sin a$$  \hspace{1cm} (7a,b)

are the boundary tractions; $a = \xi x, n$. The tilde over a symbol designates prescribed quantity. It should be noted that mixed boundary conditions could also be applied.

The prestress can be applied either before the action of the transverse load or simultaneously. In the first case, the transverse load should be applied with homogeneous in-plane boundary conditions ($u = 0$, $v = 0$) and the membrane forces in equation (2c) should be augmented by those resulting from the prestress. In this analysis, without restricting the generality, it is assumed that the membrane is prestressed by imposed boundary displacements acting simultaneously with the transverse load. Namely, the assumed boundary conditions are

$$u = \hat{u}, \quad v = \hat{v}, \quad w = \hat{w}$$ \hspace{1cm} (8a,b,c)

When the membrane is prestressed by boundary tractions, the displacements $\hat{u}, \hat{v}$ are first established by solving a plane stress problem. In any case, attention should be paid, so that the prestress results in tensile forces $N_1, N_2$ in the principal directions to avoid wrinkling of the membrane.

Substituting eqns (5) into eqns (2) and using eqns (1), we obtain the equations of motion for the heterogeneous orthogonally anisotropic membrane in terms of the displacement components

$$\left(C_1 u_{xx} + C v_{xy}\right)_x + \left(C_{12} u_{xy} + C_{12} v_{yx}\right)_y = -\left(C_1 w_{xx}^2 + \frac{C}{2} w_{yy}^2\right)_x - \left(C_{12} w_{xy} w_{yy}\right)_y$$  \hspace{1cm} (9a)

$$\left(C_2 v_{yy} + C u_{yx}\right)_y + \left(C_{12} u_{xy} + C_{12} v_{yx}\right)_x = -\left(C_2 w_{yy}^2 + \frac{C}{2} w_{xx}^2\right)_y - \left(C_{12} w_{xy} w_{xx}\right)_x$$  \hspace{1cm} (9b)

$$\rho w_{yy} - \left[C_1 \left(w_{xx} + \frac{1}{2} w_{xy}^2\right) + C \left(v_{yy} + \frac{1}{2} w_{yy}^2\right)\right] w_{xx} - \frac{1}{2} \left[C_{12} (u_{xy} + v_{yx} + w_{xx} + w_{yy})\right] w_{xy} - \frac{1}{2} \left[C_2 v_{xy} + \frac{1}{2} w_{xy}^2\right] w_{yy} = g$$  \hspace{1cm} (9c)

### 3 The analog equation method

The initial boundary value problem described by eqns (9), (8) and (4) is solved using the Analog Equation Method (AEM) following the procedure employed
for nonlinear homogeneous membranes [5]. This method is applied to the problem at hand as follows.

Let \( u = \varphi(x, y, t) \), \( v = \psi(x, y, t) \) and \( w = \omega(x, y, t) \) be the sought solution of eqns (9). These functions are twice differentiable in \( \Omega \). Thus, applying the Laplace operator to them yields

\[
\nabla^2 u_i = b_i(x, y, t) \quad (i = 1, 2, 3)
\]

where \( b_i \) are fictitious sources depending also on time. Eqns (10) are quasi-static, that is the time variable appears as a parameter. Note that here and in what it follows \( u_1, u_2 \) and \( u_3 \) stand for the functions \( u, v \) and \( w \), respectively. The fictitious sources are established using BEM. Eqns (10), which henceforth will be referred to as the analog equations of the problem at hand, indicate that the solution of eqns (9) could be established by solving these Poisson’s equations under the boundary conditions (8), if the fictitious sources \( b_i \) \((i = 1, 2, 3)\) were known.

The fictitious sources are established using BEM. Following the idea of Nardini and Brebbia [7], \( b_i \) are approximated by

\[
b_i = \sum_{j=1}^{M} a_i^{(j)} f_j
\]

where \( f_j = f_j(r) \) are \( M \) approximation radial base functions and \( a_i^{(j)} = a_i^{(j)}(t) \) are \( 3M \) coefficients to be determined. Note that \( r = r_{jp} = |P - P_j| \) is the distance between the collocation point \( P_j : \{x_j, y_j\} \) and any point \( P : \{x, y\} \in \Omega \bigcup \Gamma \).

![Boundary discretization and domain nodal points](image)

**Fig. 1:** Boundary discretization and domain nodal points

We look for a solution of the form \( \bar{u}_i + u_i^p \), where \( \bar{u}_i \) is the homogeneous solution and \( u_i^p \) a particular one. The particular solution is obtained as

\[
u_i^p = \sum a_i^{(j)} \bar{u}_j
\]

where \( \bar{u}_j = \bar{u}_j(r_{jp}) = \bar{u}_j(x, y) \) is a particular solution of
\[ \nabla^2 \tilde{u}_j = f_j \]  

The homogeneous solution is obtained from the boundary value problem

\[ \nabla^2 \tilde{u}_i = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \tilde{u}_i = \bar{u}_i - \sum_{j=1}^{M} a^{(i)}_j \tilde{u}_j \quad \text{on} \quad \Gamma \]  

(14a,b)

Thus, writing the solution of eqn (14a) in integral form, we have

\[ c\tilde{u}_i(P, t) = -\int_{\Gamma} \Big[ u^*(r_{pq}) \bar{u}_i(q, t) - \bar{u}_i(q, t) u^*_m(r_{pq}) \Big] ds_q \]  

(15)

where \( r_{pq} = |q - P| \) is the distance between the field point \( P : \{x, y\} \in \Omega \cup \Gamma \) and source point \( q : \{\xi, \eta\} \) which varies during the integration. Moreover, \( u^*(r_{pq}) = \text{ln}(r_{pq})/2\pi \) is the fundamental solution of the Laplace equation and \( u^*_m(r_{pq}) = \cos \phi / 2\pi r_{pq} \) is its normal derivative with respect to point \( q \in \Gamma \); \( \phi = \angle r_{pq}, n \). Finally, \( c = 1, \alpha/2\pi, 0 \) depending on whether \( P \in \Omega \), \( P \in \Gamma \), \( P \notin \Omega \cup \Gamma \), respectively; \( \alpha \) is the angle between the tangents to the boundary at point \( P \). For points where the boundary is smooth it is \( c = 1/2 \).

On the basis of eqn (12) and (15), the solution of eqn (10) for points \( P \) inside \( \Omega \) is written as

\[ u_i(P, t) = -\int_{\Gamma} \Big[ u^*(r_{pq}) \bar{u}_i(q, t) - \bar{u}_i(q, t) u^*_m(r_{pq}) \Big] ds_q + \sum_{j=1}^{M} a^{(i)}_j (t \tilde{u}_j(r_{pq})) \]  

(16)

It is apparent that the displacements \( u_i \) defined by eqn (15) are functions of \( x, y \). The first and second derivatives of the displacements for points inside \( \Omega \) are obtained by direct differentiation of eqn (16). Thus, we have

\[ u_{i, jk}(P, t) = -\int_{\Gamma} (u_{ik} \bar{u}_i(q, t) - \bar{u}_i u_{jk}) ds + \sum_{j=1}^{M} a^{(i)}_j \tilde{u}_{j, ik} \quad (k = 1, 2) \]  

(17)

\[ u_{i, kl}(P, t) = -\int_{\Gamma} (u_{ik} \bar{u}_i(q, t) - \bar{u}_i u_{kl}) ds + \sum_{j=1}^{M} a^{(i)}_j \tilde{u}_{j, kl} \quad (k, l = 1, 2) \]  

(18)

For reasons of conciseness, the arguments have been dropped from the above expressions.

Using BEM with \( N \) constant elements, discretizing eqn (15) and applying it to the \( N \) boundary nodal points yields

\[ \mathbf{C} \tilde{u}_i = \mathbf{H}_i - \mathbf{G}_i \]  

(19)

with \( \mathbf{C} \) being a \( N \times N \) diagonal matrix including the values of the coefficient \( c \) at the \( N \) nodal points on the boundary and \( \mathbf{H}, \mathbf{G} \) are \( N \times N \) known influence matrices originating from the integration of the kernels on the boundary elements.
Eqns (16) through (18) when discretized and applied to the $M$ nodal points inside $\Omega$ (Fig. 1) give after elimination of $\bar{u}_i$ and $\bar{u}_{i,m}$ by virtue of the boundary conditions (14b)

$$u_i = Da_i + E\bar{u}_i$$
(20a)

$$u_{i,k} = Dk_0a_i + E_k\bar{u}_i$$
(20b)

$$u_{i,kl} = D_{kl}a_i + E_{kl}\bar{u}_i$$
(20d)

where $(i = 1, 2, 3; k,l = 1, 2)$ and $D, E, \ldots, E_{kl}$ are known matrices.

Finally substituting eqns (20) into eqns (9) yields the following system of equations for $a_i$

$$a_1 = S_1(a_3)$$
(21a)

$$a_2 = S_2(a_3)$$
(21b)

$$Ma_3 - S(a_3) = g(t)$$
(21c)

where $M$ is the known $M \times M$ generalized mass matrix.

Eqn (21c) is the semi-discretized nonlinear equation of motion, which can be solved numerically to yield $a_3(t)$. Subsequently, this vector can be used in eqns (21a) and (21b) to evaluate $a_1$ and $a_2$. Once the vectors $a_i(t)$ have been computed the displacements vectors $u_i(t)$ and their derivatives at any instant $t$ are computed from eqns (20).

4 Numerical Examples

On the basis of the numerical procedure presented in the previous section a FORTRAN code has been written and numerical results for certain membranes have been obtained, which illustrate the applicability, effectiveness and accuracy of the method. The employed approximation functions $f_j$ are the multiquadrics, which are defined as

$$f_j = \sqrt{r^2 + c^2}$$
(22)

where $c$ is an arbitrary constant. Using these radial base functions the particular solution of eqn (13) is obtained as

$$\bar{u}_j = \frac{c^3}{3} \left( a_3 (r^2 + c^2) + \frac{1}{9} (r^2 + 4c^2) \sqrt{r^2 + c^2} \right)$$
(23)

In obtaining the results it was found that $c = 1$ was the optimum the value.

4.1 Square heterogeneous membrane

A square heterogeneous isotropic ($\lambda = 1$) membrane, subjected to a uniform load, has been analyzed ($N = 100$, $M = 49$). The membrane was prestressed as
shown in Fig. 2. The employed data are \( a = 2.5 \, m \), \( \rho / h = 7850 \, kgr / m^3 \), \( h = 0.002 \, m \), \( C_1 = C_2 = \bar{C}(x, y) \), \( C_{12} = (1 - \nu) \bar{C}(x, y) / 2 \), \( C = \nu \bar{C}(x, y) \), \( E = 110000 \, kN / m^2 \), \( \nu = 0.3 \). Two cases of stiffness variation have been studied (i) \( \bar{C}(x, y) = 30 C_0 (9 r^2 / a^2 + 1) / 21 \) and (ii) \( \bar{C}(x, y) = 10 C_0 \). In both cases the total stiffness of the membrane has been kept unchanged, that is \( \int_{\Omega} \bar{C}(x, y) d\Omega = 40 C_0 a^2 \). In Fig. 3 results for the natural vibrations for \( w(x, y, 0) = 0.4 \sin(\pi x / a) \cos(\pi y / a) \) and \( \dot{w}(x, y, 0) = 0 \). Moreover, in Fig. 4, the dependence of the period on the maximum amplitude is shown. It is worth noting that the variation of the ratios \( T / T_0 \), \( T_0 \) is the respective period of the linear vibration, differs negligibly.

4.2 Membrane of arbitrary shape

In this example, the heterogeneous orthotropic membrane of arbitrary shape shown in Fig. 5 was analyzed (\( N = 100 \), \( M = 49 \)). Its boundary is defined by the curve \( r = a (\sin \theta)^3 + (\cos \theta)^3 \), \( 0 \leq \theta \leq 2 \pi \). The membrane is prestressed by \( u_n = 0.04 \, m \) in the direction normal to the boundary while \( u_t = 0 \) in the tangential direction. The employed data are \( a = 5.0 \, m \), \( h = 0.002 \, m \), \( \rho / h = 5000 \, kgr / m^3 \), \( E_1 = E / \sqrt{\lambda} \), \( E_2 = E \sqrt{\lambda} \), \( \nu_1 = 0.3 \), \( \nu_2 = \lambda \nu_1 \) and \( G = E / 2(1 + \nu_1 \sqrt{\lambda}) \) where \( E = 110000 + k r^2 \), \( r = (x^2 + y^2)^{1/2} \).
and $k$ constant. In Fig. 6 results for natural vibrations are presented for $\lambda = 1$ and $k = 0$ as compared with those given in Ref. [5]. The employed initial conditions are $w(x, y, 0) = \text{deflection surface produced by the static load} \ g_0 = 0.242kN/m^2$ and $\dot{w}(x, y, 0) = 0$. Moreover, the time histories of $\ddot{w}(t) = w(0, 0, t)/w_{af}$ and $\ddot{N}_x = N_x(0, 0, t)/(N_x)_{af}$ at the center of the membrane for various values of $k$ and $\lambda$ are shown in Fig. 7 and Fig. 8, when $\dot{w}(x, y, 0) = \text{deflection surface produced by the static load} \ g_0 = 3kN/m^2$ and $\dot{w}(x, y, 0) = 0$. Finally, the forced vibrations have been studied under the so-called “static” load $g = g_{af} t_1$ for $0 \leq t \leq t_1$ and $g = g_0$ for $t_1 \leq t$ ($t_1 = 10\text{sec}$) with zero initial conditions. The time histories of $R(t) = w(0, 0, t)/w_{af}$ and $\ddot{N}_x(t) = N_x(0, 0, t)/(N_x)_{af}$ for various values of $k$ and $\lambda$ are shown in Fig. 9 and Fig. 10. Note that $w_{af}$ and $(N_x)_{af}$ designate the central deflection and membrane force produced by the static load $g_0$. 

Fig. 5: Membrane and internal nodes

Fig. 6: Variation of central deflection

Fig. 7: Time history of $\ddot{w}(t)$

Fig. 8: Time history of $\ddot{N}_x(t)$

Fig. 9: Response ratio $R(t)$

Fig. 10: Time history of $\ddot{N}_x(t)$
5 Conclusions

In this paper a boundary-only method has been presented for large deflection dynamic analysis of initially flat heterogeneous orthotropic elastic membranes. The method is based on the concept of the analog equation. From the presented analysis and the numerical examples the following main conclusions can be drawn. (a) As the method is boundary-only it has all the advantages of the BEM, i.e. the discretization and integration are performed only on the boundary. (b) The deflections and the stress resultants are computed at any point using the respective integral representation as mathematical formulas. (c) Accurate numerical results for the displacements and the stress resultants are obtained using multiquadrics with $c=1$. (d) The concept of the analog equation in conjunction with radial base functions approximation of the fictitious sources renders BEM a versatile computational method for solving difficult nonlinear dynamic problems in engineering.

6 References


