The Domain Decomposition Method for Nonlinear Analysis of Elastic Space Membranes

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Summary

In this paper the Domain Decomposition Method (DDM) is applied to the nonlinear analysis of elastic space membranes. According to this method, the domain is decomposed into non-overlapping subdomains and the boundary conditions, in the resulting virtual boundaries, are assumed. The membrane problem is solved sequentially in each subdomain and the boundary conditions are modified appropriately. The procedure is repeated until convergence is achieved. The membrane problem in each subdomain is solved using the Analog Equation Method (AEM). Example problems are presented which illustrate the method and demonstrate its efficiency and accuracy.

Introduction

The Domain Decomposition Method (DDM) is a very efficient computational tool which is used in nowadays for the solution of a great variety of problems in applied science and engineering. The rapid development of the method is due to its ability not only to reduce significantly the computational cost of the problem but also to treat domains of irregular shape, multiply connected domains, long-shaped domains, as well as domains consisting of different materials. The DDM can be implemented either in a sequential or a parallel environment, collaborating with both FEM and BEM. Moreover, the DDM can be ideally applied to the coupling of the two numerical methods.

In this paper a sequential non-overlapping DDM is applied to large deflection analysis of elastic space membranes with reference to the minimal surface [1]. According to the method the domain is decomposed into subdomains. Boundary conditions of Dirichlet type are assumed on the resulting virtual boundaries and the membrane problem is solved separately in each subdomain. Taking into account the numerical results, the boundary conditions are modified using Uzawa’s method and the membrane problem in each subdomain is solved again. The procedure is repeated until all continuity equations are fulfilled. The membrane problem in each subdomain is solved using the AEM [2]. According to this method the three coupled nonlinear equations are replaced by three uncoupled Poisson’s equations with fictitious sources under the same boundary conditions. Subsequently, the fictitious sources are established using a procedure based on the BEM and the displacement components as well as the stress resultants are evaluated at any point of the membrane from their integral representations. The developed method is boundary-only in the sense that the discretization and integration are limited only to the boundary. Thus, the method maintains all the advantages of the pure BEM. The solution is complete, that is it includes the establishment of the reference shape (minimal surface), the deformed shape under combined prestress and self-weight, and the final deformed shape under the in-service loading, which is three-dimensional. Several membranes are analyzed to illustrate the merits of the proposed method and its capabilities.

The membrane problem and the AEM solution

Consider a thin flexible elastic space (non-flat) membrane whose reference configuration is represented by the surface \( S \). The membrane may have \( K \) holes and, thus, is bounded by the \( K + 1 \)
space curves \( C_0, C_1, \ldots, C_K \). The surface \( S \) can be determined by the position vector \( r = r(\theta_1, \theta_2) \) (see Fig. 1), referred to the origin of coordinates \( O \), in terms of the general curvilinear coordinates \( \theta_1, \theta_2 \).

The covariant base vector system \( a_1, a_2 \) and contravariant one \( a^1, a^2 \) of the surface \( S \) as well as the Cartesian base vectors \( e_1, e_2, e_3 \) are shown in Fig. 1. The above representation of the surface is rather unusual for the case of the arbitrarily shaped membranes. Hence, it is more convenient to use plane coordinates \([3]\).

Consider a net of coordinate curves on the surface \( S \) and form a parallel projection of this net on to the \( xy \) plane (Fig. 1), creating the domain \( \Omega \) and the \( K + 1 \) curves \( \Gamma_0, \Gamma_1, \ldots, \Gamma_K \), which bound it. The distance between two corresponding points on \( R \) and \( \Omega \) is designated by \( z = z(x, y) \), which also represents the initial surface of the membrane.

Fig. 1: Element of the surface \( S \) and its projection on the plane \( \Omega \)

The vanishing of the first variation of the total potential of the membrane yields the following partial differential equations in terms of the displacements, which govern the equilibrium of the membrane \([4]\)

\[
(C_{11}(u_x + z_x w_x + \frac{1}{2} w_x^2) + C_{12}(v_y + z_y w_y + \frac{1}{2} w_y^2) + C_{13}(u_y + v_x + z_z w_x + z_y w_y + w_x w_y))_{,x} + \\
(C_{13}(u_x + z_x w_x + \frac{1}{2} w_x^2) + C_{23}(v_y + z_y w_y + \frac{1}{2} w_y^2) + C_{33}(u_y + v_x + z_x w_x + z_y w_y + w_x w_y))_{,y} = -p_x
\]

\[
(C_{12}(u_x + z_x w_x + \frac{1}{2} w_x^2) + C_{22}(v_y + z_y w_y + \frac{1}{2} w_y^2) + C_{23}(u_y + v_x + z_x w_x + z_y w_y + w_x w_y))_{,x} + \\
(C_{22}(u_x + z_x w_x + \frac{1}{2} w_x^2) + C_{22}(v_y + z_y w_y + \frac{1}{2} w_y^2) + C_{33}(u_y + v_x + z_x w_x + z_y w_y + w_x w_y))_{,y} = -p_y
\]

\[
(C_{11}(u_y + v_x + z_y w_x + \frac{1}{2} w_x^2) + C_{12}(v_y + z_y w_y + \frac{1}{2} w_y^2) + C_{13}(u_y + v_x + z_y w_x + z_y w_y + w_x w_y))_{,x} + \\
2[C_{12}(u_y + v_x + z_y w_x + \frac{1}{2} w_x^2) + C_{22}(v_y + z_y w_y + \frac{1}{2} w_y^2) + C_{23}(u_y + v_x + z_y w_x + z_y w_y + w_x w_y)]_{,y} = -p_x + p_y (w_x + z_x) + p_y (w_y + z_y)
\]
in $\Omega$, where $u = u(x,y)$, $v = v(x,y)$ and $w = w(x,y)$ are the displacement components of the membrane in plane coordinates; $p_x$, $p_y$, $p_z$ are the components of the load acting on the membrane and $C_{ij}$ ($i,j = 1,2,3$) are the position dependent coefficients characterizing the stiffness of the membrane [4]. Herein, the indicated derivatives are not performed for the conciseness of the expressions. The pertinent boundary conditions are

$$T_i = T_i \quad \text{or} \quad u = \tilde{u}, \quad T_y = T_y \quad \text{or} \quad v = \tilde{v}, \quad T_z (w_{x,z} + z_{,x}) + T_{y,z} (w_{y,z} + z_{,y}) = \tilde{T}_z \quad \text{or} \quad w = \tilde{w} \quad (2a,b,c)$$

on $\Gamma$, where $T_x, T_y$ are the components of the boundary tractions in plane coordinates given as

$$T_i = N_x \cos a + N_{xy} \sin a, \quad T_y = N_{xy} \cos a + N_y \sin a, \quad a = \angle x, n \quad (3a,b)$$

The membrane forces are given as

$$N_x = C_{11} \varepsilon_x + C_{12} \varepsilon_y + C_{13} \gamma_{xy}, \quad N_y = C_{12} \varepsilon_x + C_{22} \varepsilon_y + C_{23} \gamma_{xy}$$

$$N_{xy} = C_{13} \varepsilon_x + C_{23} \varepsilon_y + C_{33} \gamma_{xy} \quad (4a,b,c)$$

where

$$\varepsilon_x = \varepsilon_{11} = u_{,x} + z_{,x} w_{,x} + \frac{1}{2} w_{,2}^2, \quad \varepsilon_y = \varepsilon_{22} = v_{,y} + z_{,y} w_{,y} + \frac{1}{2} w_{,2}^2$$

$$\gamma_{xy} = 2 \varepsilon_{12} = u_{,y} + v_{,x} + z_{,x} w_{,y} + z_{,y} w_{,x} + w_{,x} w_{,y} \quad (5a,b)$$

are the strain components. The tilde over a symbol denotes prescribed quantity. In this analysis, without restricting the generality, it is assumed that the membrane is prestressed by imposed boundary displacements. Namely, the assumed boundary conditions are

$$u = \tilde{u}, \quad v = \tilde{v}, \quad w = \tilde{w} \quad (6a,b,c)$$

The boundary value problem described by the three nonlinear eqns (1) under the boundary conditions (6) is solved using the AEM. Detailed description of this method for large deflection analysis of membranes is presented in [2].

According to the concept of the analog equation, eqns (1) are replaced with three Poisson’s equations

$$\nabla^2 u_i = b_i(x,y) \quad (i = 1,2,3) \quad (7)$$

where $b_i = b_i(x_1, x_2)$ are fictitious sources. Note that $u_1, u_2, u_3$ stand here for the functions $u, v, w$, respectively. The fictitious sources are established by expressing them in radial basis function series, as described in [2], and are approximated by

$$b_i = a_i^{(1)} f_1 + a_i^{(2)} f_2 + \cdots + a_i^{(M)} f_M \quad (8)$$

where $f_i$ are $M$ approximating radial basis functions and $a_i^{(j)}$ are $3M$ coefficients to be determined. Subsequently, using the BEM the displacements and their derivatives are expressed in terms of the coefficients $a_i^{(j)}$.

Finally, eqns (1) are applied to the $M$ points inside $\Omega$ (see Fig. 2) and substitute the involved derivatives of $u_i$ yields the following set of nonlinear algebraic equations
Eqns (9) can be solved numerically to evaluate \( a' \). Once the coefficients \( a' \) have been evaluated the displacements and their derivatives are computed from the solution of the substitute problems.

\[
\begin{align*}
a^{(1)} &= F(a^{(3)}), \\
a^{(2)} &= F_2(a^{(3)}), \\
a^{(3)} &= F_3(a^{(3)})
\end{align*}
\tag{9a,b,c}
\]

The Domain Decomposition Method

We consider the domain decomposed into \( s \) subdomains \( \Omega_1, \Omega_2, \ldots, \Omega_s \) (Fig. 3). On the resulting interfaces of the subdomains the continuity conditions of displacements and fluxes (traction) should be satisfied. These conditions for the virtual boundary \( \Gamma_{mn} \) between \( \Omega_m \) and \( \Omega_n \), are described as

\[
\begin{align*}
(m) u_i &= (n) u_i, \\
(m) t_i + (n) t_i &= 0 \\
&\quad i = 1, 2, 3
\end{align*}
\tag{10a,b}
\]

where \( u_i \) and \( t_i \) are the displacement and traction components. In the case of uniform material distribution in the adjacent subdomains it can be proved that eqns (10) are simplified as

\[
\begin{align*}
(m) u_i &= (n) u_i, \\
(m) q_i + (n) q_i &= 0 \\
&\quad i = 1, 2, 3
\end{align*}
\tag{11a,b}
\]

That is the continuity of fluxes is replaced by the continuity of the normal derivatives.

The solution procedure is implemented using an iterative scheme. The boundary conditions on the virtual internal boundaries are assumed and the membrane problem is solved in each subdomain. If the continuity conditions (10b) are satisfied, the process is terminated. Otherwise, the boundary conditions are modified and the solution of the membrane problem in each subdomain is performed again. Three methods are known in the literature [5] as modification schemes of the assumed boundary conditions. Namely, the Uzawa’s method, the Schwarz Neumann-Neumann method and the Schwarz Dirichlet-Neumann method. In this investigation the Uzawa’s method is employed. According to this method Dirichlet conditions are assumed on the internal boundaries

\[
\begin{align*}
(m) u_i &= (m) u_i, \\
(n) u_i &= (n) u_i \\
&\quad i = 1, 2, 3
\end{align*}
\tag{12a,b}
\]

and the flux values are evaluated on these boundaries.

The assumed boundary conditions are modified as follows:

\[
\begin{align*}
(m) u_i^{k+1} &= (m) u_i^k - \lambda \left( (m) t_i^k + (n) t_i^k \right), \\
(n) u_i^{k+1} &= (n) u_i^{k+1} \\
&\quad i = 1, 2, 3
\end{align*}
\tag{13a,b}
\]
The superscript $k$ denotes the number of iterations, while $\lambda$ is a parameter given in advance. The initial guess for the displacements on the virtual internal boundaries is $0$ and the convergence is verified when

$$\text{sum}_i = \sum_K \left| (m_i)^k + (n_i)^k \right| \leq \varepsilon \quad i = 1, 2, 3$$

(14)

where $K$ is the number of the nodal points on the virtual internal boundary.

**Numerical examples**

**Hyperbolic paraboloidal membrane with square plan form**

A membrane, whose initial surface is assumed to be hyperbolic paraboloidal with square plan form $0 \leq x, y \leq a$, (Fig. 4); $a = 5.0m$, $h = 0.003m$, $E = 6000\, MPa$, $v = 0.3$, $\varepsilon = 0.001$ under the uniform load $p_c = 0.5\, MPa$ has been analyzed. The prestress is due to imposed displacements as shown in Fig. 5; $u = 0.05m$ and $w = 0$. Two cases of domain decomposition have been studied (i) no virtual internal boundary ($N = 100, M = 49$), (ii) virtual internal boundary at $x = a/2$ ($N_1 = N_2 = 80, M_1 = M_2 = 25$). The numerical results are presented in Fig. 6 and Fig. 7.

![Fig. 4. Hyperbolic paraboloidal membrane](image)

![Fig. 5. Prestress by imposed displacements](image)

![Fig. 6: Displacement profiles at $y = a/2$](image)

![Fig. 7: Convergence of solution for case (ii)](image)

**Elliptic membrane with central ring**

A membrane with elliptic plan form has been analysed. The outer boundary is supported on the space curve which is the intersection of the cylindrical surface $x^2 / a^2 + y^2 / b^2 = 1$, $z \geq 0$, $a = 3m$, $b = 2m$, and the sphere $x^2 + y^2 + z^2 = R^2$, $R = 3m$, while the inner one is supported on a central ring with $\rho = 0.75m$ at $z = 3m$. The geometry of the membrane and the distribution of the domain

![image]
nodal points are shown in Fig. 8. The membrane is prestressed by $u_o = 0.10m$ in the direction normal to the boundary, while $w = 0$ in the tangential direction. The employed data are $p_t = 2.0\, MPa$, $p_i(s,w) = 10.9\, Pa$, $h = 0.005m$, $E = 6000\, MPa$, $v = 0.3$, $\lambda = 0.05$, $\varepsilon = 0.0001$. The results for the response of the membrane are shown in Fig. 9 through Fig. 11.

Reference


