Large deflection analysis of elastic space membranes

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SUMMARY

In this paper a solution to the problem of elastic space (initially non-flat) membranes is presented. A new formulation of the governing differential equations is presented in terms of the displacements in the Cartesian co-ordinates. The reference surface of the membrane is the minimal surface. The problem is solved by direct integration of the differential equations using the analogue equation method (AEM). According to this method the three coupled non-linear partial differential equations with variable coefficients are replaced with three uncoupled equivalent linear flat membrane equations (Poisson’s equations) subjected to unknown sources under the same boundary conditions. Subsequently, the unknown sources are established using a procedure based on the BEM. The displacements as well as the stress resultants are evaluated at any point of the membrane from their integral representations of the solution of the substitute problems, which are used as mathematical formulae. Several membranes are analysed which illustrate the method and demonstrate its efficiency and accuracy as compared with analytical and existing numerical methods. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: non-linear; space membranes; large deflections; AEM; BEM

1. INTRODUCTION

During the past 30 years there has been a rapid expansion in the use of membrane structures. Certain advantages such as the lightness of the structure, which facilitates the coverage of very large spans, the ability to create architectural pleasing shapes and the facility of prefabrication proved that membranes could play an important role in engineering practice.

In the analysis of space elastic membranes we distinguish three stages, the form-finding stage, the prestress stage together with the self-weight and the in-service stage. Although the shape of a structure is known in most structural analyses, in the case of membranes it has to be determined first. This highly increases the difficulty of the problem solution. Many researchers...
have been involved in the form finding of membranes using various techniques. Among them, Argyris et al. [1] used the natural-shape-finding method, Linkwitz [2, 3] the direct and indirect approach of the force density method, Barnes used the dynamic relaxation method with kinetic damping [4], while Haug and Powell [5], Otto [6] used the minimal surface (soap bubble problem) a solution of which was presented by Brew and Lewis [7–9] and Katsikadelis and Nerantzaki [10]. In this investigation the initial form of the membrane is determined by solving the minimal surface problem as developed in Reference [10]. Starting from the minimal surface the shape of the membrane under prestress and self-weight is determined. Subsequently, the final form of the membrane is established by applying the external in-service loads on the deformed configuration of the previous step.

Generally, the membranes are non-linear in behaviour due to the fact that the zero or near-zero flexural stiffness renders them susceptible to large deformations, even under moderate loads. That is, such structures adapt their shape undergoing large deflections, in order to provide transverse components of the stress resultants to equilibrate the load. In the present analysis geometric non-linearity is considered which result in non-linear kinematic relations, while the strains are still small compared with the unity. A consequence of this is that the resulting differential equations governing the equilibrium of the membrane are coupled and non-linear.

The solution of the governing differential equations is a difficult mathematical problem. In the case of initially flat membranes the problem is less difficult and various solutions (analytical, approximate and numerical) are available in Reference [11]. The BEM has been also applied to the solution of the problem for both homogeneous isotropic [11] and heterogeneous orthotropic [12] initially flat membranes. For the analysis of space membranes analytical solutions are very few due to the complexity of the equations [6] and are restricted to axi-symmetric membranes where the problem is highly simplified as it becomes one dimensional. Therefore, the recourse to numerical methods is inevitable. The FEM has been used by many authors with various formulations. Haug and Powell [5] introduced a general theory referring to the non-orthogonal curvilinear system with application to the analysis of fabric membranes. A computer-aided design program has been used by Haber et al. [13] for the design of cable-reinforced membranes structures based on a non-linear analysis described in References [14, 15]. Barnes [16] used the dynamic relaxation method with kinetic damping for the form finding and analysis of membranes, while Fujikake et al. [17] investigated the non-linear analysis of fabric tension structures using an updated Lagrangian formulation to include the large displacements. Tabbarrok and Qin [18] presented a complete procedure for both form finding and load analysis of tension structures as a combination of membranes, cables and frames. All these investigators, however, presented their results in graphical form and hence their accuracy cannot be checked. An exception to this is the work of Noguchi et al. [19] who have employed the element-free Galerkin (EFG) method for the analysis of spatial structures.

In this paper the solution to large deflection analysis of space arbitrary shaped membranes is presented. First, using a variational approach a new formulation of the governing equations is derived in terms of the Cartesian displacements with reference to the minimal surface. Subsequently, a direct boundary-only solution is developed to solve the resulting three coupled non-linear partial differential equations with variable coefficients. The method is based on the concept of the analogue equation, according to which the three coupled non-linear differential equations are replaced by three equivalent linear flat membrane equations (Poisson’s equations) on the xy plane, subjected to unknown domain sources under the same boundary conditions.
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The unknown sources are established using a procedure based on BEM as it was developed for
the non-linear analysis of flat membranes [11, 12]. The use of Cartesian co-ordinates simplifies
the application of the AEM as it yields simple analogue equations that can be effectively treated
on the projected domain. The solution is complete, in the sense it includes the establishment of
the reference shape (minimal surface) following the procedure described in Reference [10], the
deformed shape under combined prestress and self-weight, and the final deformed shape under
the in-service three-dimensional loading. In each stage, the three displacement components
as well as their derivatives are computed from their integral representations of the substitute
problems, which are used as mathematical formulae. The stress resultants at any interior point
and the reactions at any boundary point are also evaluated. Several membranes are analysed
to illustrate the applicability, efficiency and accuracy of the method.

2. PROBLEM STATEMENT AND GOVERNING EQUATIONS

2.1. Definitions. Geometrical relations of the membrane surface

We consider a thin flexible elastic space (non-flat) membrane consisting of homogeneous linearly
elastic material, whose reference configuration in its undeformed state is represented by the
surface $S$. The membrane may have $K$ holes, and thus it is bounded by the $K + 1$ space curves
$C_0, C_1, \ldots, C_K$ with $C = \bigcup_{i=0}^{K} C_i$ (see Figure 1).

The surface $S$ is projected on the $x_1x_2$ plane, creating the domain $\Omega$, bounded by $\Gamma = \bigcup_{i=0}^{K} \Gamma_i$ with $\Gamma_i$ being the projection of $C_i$ (see Figure 1). Without restricting the generality,
we assume that there is a one–one correspondence of points of $S$ to those of $\Omega$. That is,
there are no lines perpendicular to $\Omega$ that meets the surface $S$ to more than one point. If $\mathbf{r}'$
is the position vector of a point on $\Omega$ and $\mathbf{r}$ of the corresponding point on $S$ (see Figure 2),

![Figure 1. Surface $S$ and its projection domain $\Omega$.](image)
we can write

$$\mathbf{r} = \mathbf{r}' + z(x_1, x_2)\mathbf{e}_3 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + z(x_1, x_2)\mathbf{e}_3$$  \hspace{1cm} (1)$$

where $z = z(x_1, x_2)$ is the equation of the surface $S$ in Cartesian co-ordinates. In what it follows, unless otherwise stated, the Greek indices take the values 1, 2, while the Latin indices take the values 1, 2, 3.

The covariant base vectors $\mathbf{a}_x$ of the surface $S$ and its normal vector $\mathbf{a}_3$, the covariant and the contravariant components $a_{\alpha\beta}$ and $a^{\alpha\beta}$ of the symmetric metric surface tensor of the first fundamental form of the surface $S$ are given as [20]

$$\mathbf{a}_x = \mathbf{r} , \quad \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{a_{11}a_{22} - a_{12}^2}}$$

$$a_{\alpha\beta} = \mathbf{a}_x \cdot \mathbf{a}_\beta$$

$$a^{11} = \frac{a_{22}}{a_{11}a_{22} - a_{12}^2} , \quad a^{12} = a^{21} = -\frac{a_{12}}{a_{11}a_{22} - a_{12}^2} , \quad a^{22} = \frac{a_{11}}{a_{11}a_{22} - a_{12}^2}$$

which for the surface (1) become

$$\mathbf{a}_x = \mathbf{e}_x + z\mathbf{e}_3 \quad \hspace{1cm} (2a)$$

$$\mathbf{a}_3 = \frac{\mathbf{e}_3 - z\mathbf{e}_x}{\sqrt{1 + z^2_1 + z^2_2}} \quad \hspace{1cm} (2b)$$

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\[ a_{\alpha \beta} = \delta_{\alpha \beta} + z,_{\alpha} z,_{\beta} \]  

(3)

\[ a^{11} = \frac{1 + z,_{1}^{2}}{1 + z,_{1} + z,_{2}^{2}} \]  

(4a)

\[ a^{22} = \frac{1 + z,_{1}^{2}}{1 + z,_{1}^{2} + z,_{2}^{2}} \]  

(4b)

\[ a^{12} = a^{21} = -\frac{z,_{1} z,_{2}}{1 + z,_{1}^{2} + z,_{2}^{2}} \]  

(4c)

where \( \delta_{\alpha \beta} \) is the Kronecker’s delta.

### 2.2. Strain tensor for the membrane

We assume that the starting reference surface \( S \) for each stage of the membrane analysis (minimal surface or prestress together with self-weight) is moved by the action of the external forces in its deformed configuration \( \bar{S} \). The surface \( \bar{S} \) is projected on the \( x_{1}x_{2} \) plane, creating the domain \( \Omega \). The displacement vector of a point of the membrane is given as

\[ \mathbf{u} = \bar{\mathbf{r}} - \mathbf{r} \]  

(5)

where \( \mathbf{r} \) is the position vector in the reference configuration \( S \) and \( \bar{\mathbf{r}} \) in the deformed one \( \bar{S} \).

The displacement vector may be expressed either in terms of the covariant base vectors of the surface \( S \) and its normal vector \( a_{3} \) as

\[ \mathbf{u} = u_{1} a_{1} + u_{2} a_{2} + u_{3} a_{3} \]  

(6)

or in terms of the Cartesian base vectors as

\[ \mathbf{u} = u_{1} \mathbf{e}_{1} + u_{2} \mathbf{e}_{2} + w \mathbf{e}_{3} \]  

(7)

where \( u_{1} = u_{1}(x, y) \), \( u_{2} = u_{2}(x, y) \) and \( w = w(x, y) \) are the displacement components of the membrane along the Cartesian axes \( x_{1}, x_{2} \) and \( x_{3} \).

Expressions (6) and (7) of the displacement vector lead to two different formulations of the problem. In the first case the resulting governing equations of the membrane are too complicated and difficult to manipulate as they are referred to the general curvilinear co-ordinate system \( a_{1}, a_{2}, a_{3} \) [21]. On the other hand, the second formulation results in governing equations which are simpler, as they are referred to the Cartesian co-ordinate system \( \mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3} \), and their integration can be performed on the projected domain of the surface of the membrane on the \( x_{1}x_{2} \) plane. The latter formulation is adopted in this investigation.

The covariant base vectors \( A_{x} \) of the surface \( \bar{S} \) of the deformed membrane is obtained as

\[ A_{x} = \bar{\mathbf{r}}_{,x} \]  

(8)

which by virtue of Equation (5) is written as

\[ A_{x} = \mathbf{r}_{,x} + \mathbf{u}_{,x} = a_{x} + \mathbf{u}_{,x} \]  

(9)
Inserting Equations (2a) and (7) into the above equation we have

\[
A = e_2 + z_2 e_3 + u_{\mu,2} e_\mu + w_{,2} e_3
= (\delta_{\mu,2} + u_{\mu,2}) e_\mu + (w_{,2} + z_2) e_3
\]  

(10)

and the components of the metric surface tensor of the deformed membrane are obtained as

\[
A_{22} = A_x \cdot A_\beta
= \delta_{22} + u_{x,2} + u_{\beta,2} + u_{\mu,2} u_{\mu,2} + (w_{,2} + z_2) (w_{,2} + z_2)
\]  

(11)

The membrane strain tensor is given as [22]

\[
\varepsilon_{22} = \frac{1}{2} (A_{22} - a_{22})
\]  

(12)

which using Equations (11) and (3) becomes

\[
\varepsilon_{22} = \frac{1}{2} (u_{x,2} + u_{\beta,2} + z_2 w_{,\beta} + z_2 w_{,2} + u_{\mu,2} w_{,2} + w_{,2} w_{,\beta})
\]  

(13)

In this analysis moderate large deflections are considered retaining the square of the slopes of the deflection surface, while, the strain components remain still small compared with the unity. Thus, the term \( u_{\mu,2} u_{\mu,2} \) is neglected and the membrane strain tensor in Cartesian co-ordinates is simplified as

\[
\varepsilon_{22} = \frac{1}{2} (u_{x,2} + u_{\beta,2} + z_2 w_{,\beta} + z_2 w_{,2} + w_{,2} w_{,\beta})
\]  

(14)

Hence, the components of the membrane strain tensor are given by

\[
\varepsilon_x \equiv \varepsilon_{11} = u_{,x} + z_{,x} w_{,x} + \frac{1}{2} w_{,x}^2
\]  

(15a)

\[
\varepsilon_y \equiv \varepsilon_{22} = v_{,y} + z_{,y} w_{,y} + \frac{1}{2} w_{,y}^2
\]  

(15b)

\[
\gamma_{xy} \equiv 2 \varepsilon_{12} = u_{,y} + v_{,x} + z_{,x} w_{,y} + z_{,y} w_{,x} + w_{,x} w_{,y}
\]  

(15c)

where the following notation was employed:

\[
u = u_1, \quad w = u_2, \quad w_{,x} = w_1, \quad w_{,y} = w_2, \quad z_{,x} = z_1, \quad z_{,y} = z_2
\]

(16)

2.3. Constitutive relations

The use of non-linear elastic stress–strain relations reflects in more realistic behaviour of the membrane materials of rubber-like, woven, neo-Hookean or Mooney–Rivlin type. These materials sometimes exhibit large strains due to large loading. However, in this analysis the magnitude of the applied loads produce strains which are small as compared with the unity leading to the assumption of linear stress–strain relation. The constitutive relation between the
contravariant components of the membrane force tensor \( n^{\alpha \beta} \) and the covariant components of the strain tensor \( \varepsilon_{\rho \lambda} \) for a homogeneous isotropic linearly elastic material is [20]

\[
n^{\alpha \beta} = DH^{\alpha \beta \rho \lambda} \varepsilon_{\rho \lambda} \tag{17}
\]

where

\[
H^{\alpha \beta \rho \lambda} = \frac{1}{2} \left[ (1 - \nu)(a^\alpha a^\beta + a^\beta a^\alpha) + 2\nu a^\alpha a^\beta \right] \tag{18}
\]

is the elasticity tensor and \( D = Eh/(1 - \nu^2) \) is the stiffness of the membrane with \( h \) being its thickness and \( E, \nu \) the material constants.

The membrane forces \( N_{\alpha \beta} \) in Cartesian co-ordinates are related to \( n^{\alpha \beta} \) as

\[
N_{\alpha \beta} = \sqrt{a} n^{\alpha \beta} \tag{19}
\]

Substituting Equation (18) into Equation (17) and using Equation (19) we obtain

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} \tag{20}
\]

where

\[
N_x = N_{11}, \quad N_y = N_{22}, \quad N_{xy} = N_{12}
\]

and \( C_{ij} \) are position-dependent coefficients given as

\[
\begin{align*}
C_{11} &= D \frac{(1 + z_y^2)}{(1 + z_x^2 + z_y^2)^{3/2}}, \\
C_{12} &= D \frac{z_x z_y^2 + \nu(1 + z_x^2 + z_y^2)}{(1 + z_x^2 + z_y^2)^{3/2}}, \\
C_{13} &= -D \frac{z_x z_y^2}{(1 + z_x^2 + z_y^2)^{3/2}}, \\
C_{22} &= D \frac{(1 + z_x^2)}{(1 + z_x^2 + z_y^2)^{3/2}}, \\
C_{23} &= -D \frac{z_x z_y^2}{(1 + z_x^2 + z_y^2)^{3/2}}, \\
C_{33} &= D \frac{2z_x^2 + 2z_y^2}{2(1 + z_x^2 + z_y^2)^{3/2}}, \\
C_{32} &= C_{33}
\end{align*} \tag{22}
\]

Note that for \( z_x = z_y = 0 \) Equations (22) yield the constitutive equations of the flat membrane.

2.4. Equilibrium equations of the membrane

The total potential of the deformed membrane is given as

\[
\Pi = \frac{1}{2} \int_\Omega \left( N_x \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy} \right) d\Omega - \int_\Omega (p_x u + p_y v + p_z w) d\Omega \\
- \int_\Gamma (\tilde{T}_x u + \tilde{T}_y v + \tilde{T}_z w) ds \tag{23}
\]
where $p_x, p_y, p_z$ are the Cartesian components of the load acting on the membrane; $\tilde{T}_x, \tilde{T}_y$ and $\tilde{T}_z$ are the external stretching forces acting along the boundary. The condition $\delta \Pi = 0$ yields the following differential equations, which govern the equilibrium of the membrane:

$$
N_{x,x} + N_{xy,y} + p_x = 0 \\
N_{xy,x} + N_{y,y} + p_y = 0 \\
N_x (z,xx + w,xx) + 2N_{xy}(z,xy + w,xy) + N_y (z,yy + w,yy) \\
= -p_z + p_x (z,x + w,x) + p_y (z,y + w,y)
$$

in $\Omega$, together with the boundary conditions

$$
T_x = \tilde{T}_x \quad \text{or} \quad u = \tilde{u} \\
T_y = \tilde{T}_y \quad \text{or} \quad v = \tilde{v}
$$

(25)

on $\Gamma$. The tilde over a symbol designates prescribed quantity. $T_x, T_y$ are the components of the boundary tractions in Cartesian co-ordinates given as

$$
T_x = N_x \cos \theta + N_{xy} \sin \theta \\
T_y = N_{xy} \cos \theta + N_y \sin \theta
$$

(26a)

(26b)

$\theta = \chi, \mathbf{n}$. It should be noted that mixed boundary conditions can also be applied. The boundary conditions are non-linear, when tractions are prescribed, since $T_x$ and $T_y$ depend on the squares of $w,x$ and $w,y$. Hence, using Equations (20) the traction boundary conditions become

$$
(C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\gamma_{xx}) \cos \theta + (C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\gamma_{xy}) \sin \theta = \tilde{T}_x \\
(C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\gamma_{xy}) \cos \theta + (C_{21}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\gamma_{xy}) \sin \theta = \tilde{T}_y \\
(C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\gamma_{xy}) \cos \theta (w,x + z,x) \\
+ (C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\gamma_{xy}) [\sin \theta (w,x + z,x) + \cos \theta (w,y + z,y)] \\
+ (C_{21}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\gamma_{xy}) \sin \theta (w,y + z,y) = \tilde{T}_z
$$

(27a)

(27b)

(27c)

Attention should be paid, so that in every stage of the analysis tensile forces $N_1, N_2$ in the principal directions are produced to avoid wrinkling of the membrane, i.e.

$$
N_{1,2} = \frac{N_x + N_y}{2} \pm \sqrt{\frac{(N_x - N_y)^2}{4} + (N_{xy})^2} > 0
$$

(28)
2.5. The boundary value problem of the membrane

Substituting Equations (20) into Equations (24) and using Equations (15) we obtain the equilibrium equations for the space membranes in terms of the displacement components

\[
\begin{align*}
[C_{11}(u, x + z, x w, x + \frac{1}{2} w_{x}^2) + C_{12}(v, y + z, y w, y + \frac{1}{2} w_{y}^2)]_{,x} \\
+ [C_{13}(u, y + v, x + z, x w, y + z, y w, x + w_{x} w, y)]_{,x} \\
+ [C_{13}(u, x + z, x w, x + \frac{1}{2} w_{x}^2) + C_{23}(v, y + z, y w, y + \frac{1}{2} w_{y}^2)]_{,x} \\
+ C_{33}(u, y + v, x + z, x w, y + z, y w, x + w_{x} w, y)]_{,y} = -p_x \\
\end{align*}
\]

(29a)

\[
\begin{align*}
[C_{13}(u, x + z, x w, x + \frac{1}{2} w_{x}^2) + C_{23}(v, y + z, y w, y + \frac{1}{2} w_{y}^2)]_{,y} \\
+ C_{33}(u, y + v, x + z, x w, y + z, y w, x + w_{x} w, y)]_{,y} = -p_y \\
\end{align*}
\]

(29b)

\[
\begin{align*}
[C_{11}(u, x + z, x w, x + \frac{1}{2} w_{x}^2) + C_{12}(v, y + z, y w, y + \frac{1}{2} w_{y}^2)]_{,x} \\
+ C_{13}(u, y + v, x + z, x w, y + z, y w, x + w_{x} w, y)]_{,x} \\
+ 2[C_{13}(u, x + z, x w, x + \frac{1}{2} w_{x}^2) + C_{23}(v, y + z, y w, y + \frac{1}{2} w_{y}^2)] \\
+ C_{33}(u, y + v, x + z, x w, y + z, y w, x + w_{x} w, y)]_{,xy} \\
+ [C_{12}(u, x + z, x w, x + \frac{1}{2} w_{x}^2) + C_{22}(v, y + z, y w, y + \frac{1}{2} w_{y}^2)] \\
+ C_{23}(u, y + v, x + z, x w, y + z, y w, x + w_{x} w, y)]_{,yy} \\
= -p_z + p_x(z, x + w, x) + p_y(z, y + w, y) \\
\end{align*}
\]

(29c)

in Ω. For the conciseness of the expressions the differentiations have not been performed here. However, in the subsequent numerical analysis, the differentiations have been performed using a symbolic language (MAPLE) and converted to FORTRAN in the computer program.

Equations (29) together with the boundary conditions (27), constitute the boundary value problem of the non-linear theory of membranes in the Cartesian co-ordinates. Equations (29) are reported for the first time [23].

3. THE ANALOGUE EQUATION METHOD FOR THE NON-LINEAR ANALYSIS OF SPACE MEMBRANES

The boundary value problem described by Equations (29) and (27) is solved using the analogue equation method (AEM) as developed for the non-linear analysis of flat membranes [11, 12]. This method is applied to the problem at hand as follows.
Let \( u = u(x, y) \), \( v = v(x, y) \) and \( w = w(x, y) \) be the sought solution to Equations (29). These functions are twice differentiable in \( \Omega \). Thus, if the Laplacian operator is applied to them, we have

\[
\nabla^2 u_i = b_i(x, y) \tag{30}
\]

where \( u_1 = u, \ u_2 = v \) and \( u_3 = w \).

Equations (30), which henceforth will be referred to as the analogue equations of the problem, indicate that the solution of Equations (29) could be established by solving these Poisson’s equations under the boundary conditions (27), if the unknown sources \( b_i(x, y) \) were known. Their establishment is accomplished following the procedure below.

We write the solution of Equation (30) in integral form. Thus, we have [24]

\[
c u_i(x) = \int_\Omega u^*(x, y) b_i(y) \, d\Omega_y - \int_\Gamma \left[ u^*(x, \xi) u_{i,n}(\xi) - u_i(\xi) u_{i,n}^*(x, \xi) \right] ds_\xi \tag{31}
\]

in which \( x \in \Omega \cup \Gamma, \ y \in \Omega \) and \( \xi \in \Gamma; \ u^* = \ln(r)/2\pi \) is the fundamental solution of the Laplace equation and \( u_{i,n}^* \) is its derivative normal to the boundary with \( r = ||y - x|| \) or \( r = ||\xi - x|| \) being the distance between the points \( x, y \) or \( x, \xi \); \( c = 1, \pi/2, 0 \) depending on whether \( x \in \Omega, \ x \in \Gamma, \ x \notin \Omega \cup \Gamma \), respectively; \( \pi \) is the angle between the tangents to the boundary at point \( x \). For points where the boundary is smooth it is \( c = 1/2 \).

Equation (31) when applied to boundary points yields the boundary integral equation

\[
a \frac{1}{2\pi} u_i(x) = \int_\Omega u^*(x, y) b_i(y) \, d\Omega_y - \int_\Gamma \left[ u^*(x, \xi) u_{i,n}(\xi) - u_i(\xi) u_{i,n}^*(x, \xi) \right] ds_\xi \tag{32}
\]

Equation (32) is a domain-boundary integral equation and could be solved using domain discretization to approximate the domain integrals. This, however, would spoil the advantages of the BEM over the domain methods. We can maintain the pure boundary character of the method by converting the domain integrals to boundary line integrals using the following procedure. The unknown sources functions \( b_i \) are approximated by the series

\[
b_i = \sum_{j=1}^{M} a_i^j f_j \tag{33}
\]

where \( f_j = f_j(r_{xx,j}) \) is a set of \( M \) radial basis approximation functions and \( a_i^j \) are coefficients to be determined; \( r_{xx,j} = ||x - x_j|| \) with \( x_j \) being the \( M \) collocation points located in the domain \( \Omega \) (see Figure 3). Using the Green’s reciprocal identity [24] the domain integral in Equation (32) becomes

\[
\int_\Omega u^*(x, y) b_i(y) \, d\Omega_y = \sum_{j=1}^{M} a_i^j \int_\Omega u^*(x, y) f_j(y) \, d\Omega_y
\]

\[
= \sum_{j=1}^{M} a_i^j \left\{ c\hat{u}_j(x) + \int_\Gamma \left[ u^*(x, \xi) \hat{u}_{j,n}(\xi) - \hat{u}_j(\xi) u_{j,n}^*(x, \xi) \right] ds_\xi \right\} \tag{34}
\]

in which \( \hat{u}_j(r_{xx,j}) \) is a particular solution of the equation

\[
\nabla^2 \hat{u}_j = f_j \tag{35}
\]
and \( \hat{u}_{j,n} \) its derivative normal to the boundary. Note that \( \hat{u}_j \) can be always established when \( f_j \) is specified. Taking into consideration Equation (34), Equation (32) is written as

\[
\frac{a}{2\pi} u_i(x) = \sum_{j=1}^{M} a^j_i \left\{ c\hat{u}_j(x) + \int_{\Gamma} [u^*(x, \xi)\hat{u}_{j,n}(\xi) - \hat{u}_j(\xi)u^*_{n}(x, \xi)] \, ds_\xi \right\} \\
- \int_{\Gamma} [u^*(x, \xi)u_{i,n}(\xi) - u_i(\xi)u^*_{n}(x, \xi)] \, ds_\xi
\] (36)

For points \( x \in \Omega \ (c = 1) \) the displacements can be evaluated from Equation (31), which by virtue of Equation (34) becomes

\[
u_i(x) = \sum_{j=1}^{M} a^j_i \left\{ c\hat{u}_j(x) + \int_{\Gamma} [u^*(x, \xi)\hat{u}_{j,n}(\xi) - \hat{u}_j(\xi)u^*_{n}(x, \xi)] \, ds_\xi \right\} \\
- \int_{\Gamma} [u^*(x, \xi)u_{i,n}(\xi) - u_i(\xi)u^*_{n}(x, \xi)] \, ds_\xi
\] (37)

It is apparent that the displacements \( u_i \) given by the above equation are functions of \( x, y \). Hence, the first and second derivatives of \( u_i \) obtained by direct differentiation. Namely,

\[
u_{i,kl}(x) = \sum_{j=1}^{M} a^j_i \left\{ c\hat{u}_{j,kl}(x) + \int_{\Gamma} [u^*,kl(x, \xi)\hat{u}_{j,n}(\xi) - \hat{u}_j(\xi)u^*_{n,kl}(x, \xi)] \, ds_\xi \right\} \\
- \int_{\Gamma} [u^*,kl(x, \xi)u_{i,n}(\xi) - u_i(\xi)u^*_{n,kl}(x, \xi)] \, ds_\xi
\] (38)

where \( k, l = 0, x, y \). Note that \( u_{i,00} \equiv u_i \).

Figure 3. Boundary discretization and domain nodal points.
The expressions of the derivatives of the kernel functions are given in the Appendix.

The final step of AEM is to apply Equations (29) to the \( M \) collocation points inside \( \Omega \). We, thus, obtain a set of \( 3M \) simultaneous equations of the form

\[
F_j^{(1)}\{(u_{i,x})^j, (u_{i,y})^j, (u_{i,xx})^j, (u_{i,xy})^j, (u_{i,yy})^j\} = -(p_x)^j
\]  
\[
F_j^{(2)}\{(u_{i,x})^j, (u_{i,y})^j, (u_{i,xx})^j, (u_{i,xy})^j, (u_{i,yy})^j\} = -(p_y)^j
\]  
\[
F_j^{(3)}\{(u_{i,x})^j, (u_{i,y})^j, (u_{i,xx})^j, (u_{i,xy})^j, (u_{i,yy})^j\} = -(p_z)^j
\]

in which \( j = 1, 2, \ldots, M \). Substitution of Equations (38) into Equations (39) yield a set of \( 3M \) non-linear algebraic equations which can be used to evaluate the coefficients \( a_i^j \). The AEM can be implemented only numerically using the procedure presented in the following section.

4. NUMERICAL IMPLEMENTATION OF THE AEM

The BEM with constant elements is used to approximate the boundary integrals in Equations (36)–(38). In this case it is \( a = \pi \), hence \( c = 1/2 \). If \( N \) is the number of the boundary nodal points (see Figure 3), then for the \( m \) nodal point Equation (36) is written as

\[
\frac{1}{2} u_i^m = \sum_{j=1}^{M} F_{mj} a_i^j + \sum_{k=1}^{N} \tilde{H}_{mk} u_i^k - \sum_{k=1}^{N} G_{mk} u_{i,n}^k \tag{40}
\]

where

\[
\tilde{H}_{mk} = \int_k u_{i,n}^s(r_{mk}) \, ds \tag{41}
\]

\[
G_{mk} = \int_k u^s(r_{mk}) \, ds \tag{42}
\]

\[
F_{mj} = c\tilde{u}_j^m - \sum_{k=1}^{N} \tilde{H}_{mk} \tilde{u}_j^k + \sum_{k=1}^{N} G_{mk} (\tilde{u}_{i,n})_j^k, \quad m = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, M \tag{43}
\]

In the above equations \( \int_k \) indicates integration on the \( k \) element.

Applying Equation (40) to all boundary nodal points and using matrix notation yields

\[
Hu_i - Gu_{i,n} + Fa_i = 0 \tag{44}
\]

where \( a_i = \{a_i^1, a_i^2, \ldots, a_i^M\}^T \); \( u_i, u_{i,n} \) are the vectors of the \( N \) boundary nodal values of \( u_i \), \( u_{i,n} \), respectively, and

\[
H = \tilde{H} - \frac{1}{2} I \tag{45}
\]

where \( I \) is the unit matrix.

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Moreover, using the same discretization in Equations (37) and (38) and applying them to the $M$ collocation points we obtain

$$\bar{u}_i = F_a + H u_i - G_{i,n}$$  \hspace{1cm} (46)$$

$$\bar{u}_{i,kl} = F_{kl} a_i + H_{kl} u_i - G_{kl} u_{i,n}, \hspace{1cm} k, l = 0, x, y$$  \hspace{1cm} (47)$$

In the previous expression $H, G_{kl}$ are known $M \times N$ matrices originating from the integration of the kernels on the boundary elements of Equations (37) and (38) and $\bar{u}_i, \bar{u}_{i,kl}$ are $M \times 1$ vectors including the values of $u_i$ and its derivatives at the interior collocation points. Note that the resulting line integrals are regular since the distance $r_{jm}$ in the kernels do not vanish.

Subsequently, substituting Equations (47) into Equations (39) and using Equation (44) to eliminate $u_i$ and $u_{i,n}$ yields the following system of equations for $a_i$:

$$A_{11} a_1 + A_{12} a_2 = B_1(a_3)$$  \hspace{1cm} (48a)$$

$$A_{21} a_1 + A_{22} a_2 = B_2(a_3)$$  \hspace{1cm} (48b)$$

$$A(a_1, a_2, a_3)a_3 = -p_z$$  \hspace{1cm} (48c)$$

where $A_{ij} (i, j = 1, 2)$ are $M \times M$ known constant matrices, while the matrix $A$ and the vectors $B_1, B_2$ depend on the unknown vectors $a_i$. Equations (48a) and (48b) can be solved for $a_1$ and $a_2$. Thus, Equations (48) can be reduced to

$$a_1 = S_1(a_3)$$  \hspace{1cm} (49a)$$

$$a_2 = S_2(a_3)$$  \hspace{1cm} (49b)$$

$$S(a_3)a_3 = -p_z$$  \hspace{1cm} (49c)$$

Equation (49c) constitutes a system of $M$ non-linear algebraic equations which can be solved to yield $a_3$. In this investigation a successful solution to Equation (49c) has been achieved by converting it to an equivalent minimization problem where the objective function is the sum of the squares of the $M$ non-linear algebraic equations. That is,

$$T(a_3) = \sum_{j=1}^{M} [S_j(a_3)a_3 + (p_z)_j]^2$$  \hspace{1cm} (50)$$

In each step the values of $a_1$ and $a_2$ are updated using Equations (49a) and (49b). In general, the convergence of the solution is good for a prestressed membrane, and it becomes worse with decreasing prestressing.
Subsequently, the coefficients $a_i$ are employed in Equation (44) to evaluate $u_{i,n}$ and the displacements and their derivatives at the $M$ nodal points are computed from Equations (46) and (47). For points $P \in \Omega$ not coinciding with the nodal points these quantities are evaluated from the discretized counterparts of Equations (37) and (38).

The stress resultants $N_x, N_y, N_{xy}$ at any interior collocation point are computed from Equations (20) after the substitution of Equations (47) into Equations (15).

It should be noted that traction boundary conditions can be also encountered by expressing Equations (27) in terms of the boundary slopes and displacements. In this case, however, the boundary conditions are non-linear and the related boundary equations should be included as additional non-linear equations because the boundary quantities $u_i$ and $u_{i,n}$ cannot be eliminated.

5. NUMERICAL EXAMPLES

On the basis of the numerical procedure presented in the previous section a FORTRAN code has been written and numerical results for certain membranes have been obtained, which illustrate the applicability, effectiveness and accuracy of the method. The employed approximation functions $f_j$ are the multiquadrics, which are defined as

$$f_j = \sqrt{r^2 + c^2}, \quad r = \sqrt{(x - x_j)^2 + (y - y_j)^2}, \quad j = 1, 2, \ldots, M$$

where $c$ is the shape parameter [25]. Using these radial basis functions, the particular solution of Equation (35) is obtained as

$$\hat{u}_j = -\frac{c^3}{3} \ln(\sqrt{r^2 + c^2} + c^2) + \frac{1}{9} (r^2 + 4c^2) \sqrt{r^2 + c^2}$$

The derivatives of $\hat{u}_j$ are given in the Appendix.

In the first three examples, membranes with known initial configuration are examined in order to compare the results with those obtained by an analytical (see Example 5.1), a FEM (see Examples 5.2 and 5.3) and an EFG solution [19] (see Example 5.2). In the other two examples the solution procedure is complete. Namely, it includes the establishment of (i) the reference shape (minimal surface), (ii) the deformed shape under combined prestress and self-weight and (iii) the final deformed shape under the in-service loading.

5.1. Spherical membrane

A spherical membrane with assumed initial configuration

$$z(x, y) = \pm \sqrt{R_0^2 - x^2 - y^2}, \quad x = R_0 \cos \phi, \quad y = R_0 \sin \phi, \quad 0 \leq \phi \leq 2\pi$$

loaded by internal pressure $p_0$ with zero prestress ($\ddot{u} = \dddot{v} = \dddot{w} = 0$), was analysed. The analytical solution is [6]

$$w_R = R \frac{\kappa}{1 - \kappa} \quad \text{with} \quad \kappa = \frac{p_0 R (1 - v)}{2Eh}$$
Table I. Radial deflection of the membrane of Example 5.1.

<table>
<thead>
<tr>
<th>$p_0$</th>
<th>AEM</th>
<th>Analytical [6]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.035</td>
<td>0.036</td>
</tr>
<tr>
<td>2.0</td>
<td>0.072</td>
<td>0.075</td>
</tr>
<tr>
<td>3.0</td>
<td>0.113</td>
<td>0.117</td>
</tr>
</tbody>
</table>

Figure 4. Double parabolic membrane with square plan form.

and it is used to test the accuracy of the proposed method. The employed data are $R_0 = 1.0 \text{m}$, $Eh = 10 \text{kN/m}$, and $\nu = 0.3$. Due to the symmetry, only a quarter of the spherical surface was analysed. The results were obtained with $N = 91$ and $M = 38$. The radial deflection of the membrane versus the load $p_0$ is shown in Table I.

5.2. Double parabolic membrane with square plan form

A membrane with square plan form and assumed initial configuration given by the equation

$$z(x, y) = \frac{16f}{a^2b^2} \left( \frac{x - a}{2} \right)^2 \left( \frac{y - b}{2} \right)^2$$

was analysed (see Figure 4). The employed data are $a = b = 1.0 \text{m}$, $f = 0.1 \text{m}$, $h = 0.001 \text{m}$, $E = 6000 \text{MPa}$ and $\nu = 0.267$. First, the membrane was analysed with zero prestress ($\bar{u} = \bar{v} = \bar{w} = 0$) in order to compare the results with those obtained by FEM using the NASTRAN code with 200 triangular plate elements and the EFG Method [19]. The results were obtained with $N = 100$ and $M = 49$. The central deflection versus the load $p_z$ are shown in Figure 5 and Table II. Additional results for the response of the membrane are presented in Figures 6 and 7 for $p_z = 1.5 \text{MPa}$. The discrepancy of the results between AEM and FEM, EFG which appear closer to each other can be explained from the fact that the latter methods use the strain model and the stress–strain relations of the plane elasticity neglecting the curvature of the membrane.
This discrepancy was also reported in the non-linear analysis of flat membranes [11] where the AEM results were found more accurate than the FEM ones as compared with existing analytical solutions. The membrane was also analysed under prestress by imposed boundary displacements ($\tilde{u} = \tilde{v} = 0.01 \text{ m}$) as shown in Figure 8 and subjected to a uniform load. The obtained results for the central deflection versus the load $p_z$ are shown in Figure 9 and Table III while the profiles of the displacements at $y = 0$ are presented in Figure 10 for $p_z = 3.5 \text{ MPa}$.

5.3. Hyperbolic paraboloidal membrane with square plan form

A membrane with square plan form and assumed initial configuration given by the equation

$$z(x, y) = \frac{\Delta f}{ab} xy + \frac{C_2}{a} x + \frac{C_4}{b} y$$

was analysed; $\Delta f = C_3 - C_2 - C_4$ (see Figure 11). The employed data are $a = b = 1.0 \text{ m}$, $C_3 = 0$, $C_2 = C_4 = 0.3 \text{ m}$, $h = 0.003 \text{ m}$, $E = 6000 \text{ MPa}$ and $\nu = 0.3$. First, the membrane was analysed with zero prestress ($\tilde{u} = \tilde{v} = \tilde{w} = 0$) in order to compare the results with those obtained by
Figure 6. Profiles of $u$ and $w$ at $y = 0$ in the membrane of Example 5.2 under zero prestress.

Figure 7. Variation of $N_x$, $N_y$ at $x = 0$ in the membrane of Example 5.2 under zero prestress.

FEM using the NASTRAN code with 200 triangular plate elements. The results were obtained with $N = 100$ and $M = 49$. The central deflection versus the load $p_z$ are shown in Figure 12 and Table IV. Additional results for the response of the membrane are presented in Figures 13 and 14 for $p_z = 3.0$ MPa. The membrane was also analysed under prestress by imposed boundary displacements ($\bar{u} = \bar{v} = 0.01$ m) as shown in Figure 8 and subjected to a uniform load. The obtained results for the central deflection versus the load $p_z$ are shown in Figure 15 and Table V while the profiles of the displacements at $y = 0$ are presented in Figure 16 for $p_z = 4.5$ MPa.
Figure 8. Prestress by imposed displacements.

Figure 9. Central deflection versus load in the prestressed membrane of Example 5.2.

Table III. Central deflection of the prestressed membrane of Example 5.2.

<table>
<thead>
<tr>
<th>$p_0$ (MPa)</th>
<th>$w_0$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.40</td>
<td>0.08297</td>
</tr>
<tr>
<td>2.10</td>
<td>0.10922</td>
</tr>
<tr>
<td>2.80</td>
<td>0.13012</td>
</tr>
<tr>
<td>3.50</td>
<td>0.14774</td>
</tr>
</tbody>
</table>
5.4. Membrane with rhombus plan form

In this example a membrane with rhombus plan form shown in Figure 17, was analysed. The geometry of the domain as well as the distribution of the employed internal nodal points are presented in Figure 18. The membrane is prestressed by \( u_p = 0.09 \text{ m} \) in the direction normal to the boundary, while \( u_t = 0 \) in the tangential direction. The employed data are \( E = 1.1 \times 10^5 \text{kN/m}^2, \gamma = 0.3, h = 0.002 \text{m}, a_1 = 14 \text{m}, a_2 = 18 \text{m}, h_1 = 8 \text{m}. \) The results were obtained with \( N = 100 \) and \( M = 41. \) The central deflection versus the load \( p_z \) for various values of the ratio \( h_2/h_1 \) are shown in Figure 19, while the variation of the central deflection versus the ratio \( h_2/h_1 \) for \( p_z = 7.5 \text{kN/m}^2 \) is presented in Figure 20. Moreover, the deflected cross-section profile \( z \) of the membrane at \( y = 0 \) for \( h_2/h_1 = 3/8, 5/8, 7/8 \) are shown in Figures 21–23.
respectively, for the three different stages of the analysis (self-weight 10 N/m², in-service load 7.5 kN/m²).

5.5. Membrane with elliptic plan form

In this example a membrane with elliptic plan form was analysed. The geometry of the domain as well as the distribution of the employed internal nodal points are presented in Figure 24. The membrane is supported on the space curve, which is the intersection of the cylindrical surface \( x^2/a^2 + y^2/b^2 = 1 \) and the sphere \( x^2 + y^2 + z^2 = R^2, \ z \geq 0 \) while it is prestressed by \( u_n = 0.05 \text{ m} \) in the direction normal to the boundary and \( u_t = 0 \) in the tangential direction. The employed data are \( E = 1.1 \times 10^5 \text{ kN/m}^2 \), \( v = 0.3 \), \( h = 0.002 \text{ m} \), \( a = 1.5 \text{ m} \), \( b = 1 \text{ m} \) and \( R = 5 \text{ m} \). The results were obtained with \( N = 100 \) and \( M = 49 \). The reference shape, the central deflection versus the load \( p_z \) and the final deformed shape of the membrane for \( p_z = 40 \text{ kN/m}^2 \) are shown in Figures 25–27, respectively. Moreover, the deflected cross-section profile \( z \) of the membrane at \( y = 0 \) and \( x = 0 \) are presented in Figures 28 and 29 for the three different stages of the analysis (self-weight 10 N/m², in-service load 40 kN/m²).
Figure 13. Profiles of $u$ and $w$ at $y=0$ in the membrane of Example 5.3 under zero prestress.

Figure 14. Variation of $N_x$, $N_y$ at $x=0$ in the membrane of Example 5.3 under zero prestress.

6. CONCLUSIONS

From the presented analysis and the numerical examples the following main conclusions can be drawn:

(a) A BEM method is developed for the non-linear analysis of membranes.
Figure 15. Central deflection versus load in the prestressed membrane of Example 5.3.

Table V. Central deflection of the prestressed membrane of Example 5.3.

<table>
<thead>
<tr>
<th>$p_z$ (MPa)</th>
<th>$w_0$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>0.09309</td>
</tr>
<tr>
<td>2.7</td>
<td>0.11684</td>
</tr>
<tr>
<td>3.6</td>
<td>0.13516</td>
</tr>
<tr>
<td>4.5</td>
<td>0.15025</td>
</tr>
</tbody>
</table>

Figure 16. Profiles of $u$ and $w$ at $y=0$ in the prestressed membrane of Example 5.3.
Figure 17. Membrane with rhombus plan form.

Figure 18. Geometry of the domain and distribution of the internal nodal points in the membrane of Example 5.4.

(b) The solution procedure is complete, i.e. form finding, prestress together with self-weight and in-service loading.
(c) The starting point is the derivation of the non-linear equations describing the response of the membrane in terms of the Cartesian displacements, which are reported for the first time.
Figure 19. Central deflection versus the load $p_z$ in the membrane of Example 5.4.

Figure 20. Variation of the central deflection versus the ratio $h_2/h_1$ in the membrane of Example 5.4.
Figure 21. Deflected cross-section profile of the membrane of Example 5.4 at $y = 0$ for $h_2/h_1 = 3/8$.

Figure 22. Deflected cross-section profile of the membrane of Example 5.4 at $y = 0$ for $h_2/h_1 = 5/8$.

(d) These equations are suitable to an AEM solution by converting them to three linear flat membrane equations (Poisson’s equations). Besides its effectiveness, the method is boundary-only in the sense that the discretization and integration are performed only on the boundary.
Figure 23. Deflected cross-section profile of the membrane of Example 5.4 at \( y = 0 \) for \( h_2/h_1 = 7/8 \).

Figure 24. Geometry of the domain and distribution of the internal nodal points in the membrane of Example 5.5.
Figure 25. Reference shape of the membrane of Example 5.5 (minimal surface).

Figure 26. Central deflection versus the load $p_z$ in the membrane of Example 5.5.

(e) The deflections and the stress resultants are computed at any point using the respective integral representations, of the substitute problems, as mathematical formulae.

(f) The accuracy of the results is very good as compared with the analytic solution.
Figure 27. Final deformed shape of the membrane of Example 5.5 (prestress, self-weight and in-service loading).

Figure 28. Deflected cross-section profile of the membrane of Example 5.5 at $y = 0$.

(g) The discrepancy of the results between the AEM and the other numerical methods (FEM, EFG) which appear to be close to each other can be explained from the different strain models and stress–strain relations. FEM and EFG use the strain model
and the stress–strain relations of the plane elasticity neglecting the curvature of the membrane.

(h) Concerning the numerical cost the method is computationally effective as compared to existing numerical solutions. In general, the convergence of the solution is good for a prestressed membrane, and it becomes worse with decreasing prestressing. For prestressed membranes a few seconds are enough on a PC to get accurate results. It is noted that the numerical results are not sensitive to the material data.

(i) For the illustration of the method certain simplifications have been made which will be addressed in a future work. More specifically, the use of non-linear elastic stress–strain relations which reflects more realistic behaviour of the employed material (rubber-like, woven, neo-Hookean, Mooney–Rivlin), the use of traction boundary conditions, as well as, the solution of the cable supported membrane problem are the subject of future works.

APPENDIX A

A.1. Derivatives of the kernel functions

The derivatives of the distance

\[ r = \sqrt{(\xi - x)^2 + (\eta - y)^2} \quad \{x, y\} \in \Omega, \quad \{\xi, \eta\} \in \Gamma \]  

(A1)
are evaluated from the following relations:

\[ r_x = - \xi = - \frac{\xi - x}{r}, \quad r_y = - \eta = - \frac{\eta - y}{r} \]

\[ r_{xx} = \frac{r^2_y}{r}, \quad r_{yy} = \frac{r^2_x}{r}, \quad r_{xy} = - \frac{r_x r_y}{r} \]  \hspace{1cm} (A2)

\[ r_n = -(r_x n_\xi + r_y n_\eta), \quad r_t = - (r_x n_\eta + r_y n_\xi) \]

\( n_\xi, n_\eta \) are directional cosines of the outward normal vector to the boundary at point \( \{\xi, \eta\} \).

Using Equations (A2) the derivatives of the fundamental solution may be expressed as

\[ u^*_x = \frac{1}{2\pi} \frac{r_x}{r}, \quad u^*_y = \frac{1}{2\pi} \frac{r_y}{r} \]

\[ u^*_{xx} = \frac{1}{2\pi} \frac{r^2 - r_y^2}{r^2}, \quad u^*_{yy} = - u^*_{xx}, \quad u^*_{xy} = - \frac{1}{\pi} \frac{r_x r_y}{r^2} \]

\[ u^*_{nxx} = - \frac{1}{\pi} \frac{(r_y^2 - r_x^2) r_n + 2 r_x r_y r_t}{r^3} \]

\[ u^*_{nyy} = - u^*_{nxx} \]

\[ u^*_{nxy} = - \frac{1}{\pi} \frac{(r_y^2 - r_x^2) r_t - 2 r_x r_y r_n}{r^3} \]  \hspace{1cm} (A3)

A2. Derivatives of the function \( \hat{u}_j \)

Differentiating Equation (52) gives

\[ \hat{u}_{j,x} = \frac{1}{3\sqrt{r^2 + c^2}} \left( r^2 + 2c^2 - \frac{c^3}{\sqrt{r^2 + c^2} + c} \right) (x - x_j) \]

\[ \hat{u}_{j,y} = \frac{1}{3\sqrt{r^2 + c^2}} \left( r^2 + 2c^2 - \frac{c^3}{\sqrt{r^2 + c^2} + c} \right) (y - y_j) \]

\[ \hat{u}_{j,xx} = \frac{1}{3(r^2 + c^2)^{3/2}} \left[ r^2 + c^2 - \frac{2\sqrt{r^2 + c^2} + c}{(\sqrt{r^2 + c^2} + c)^2} \right] (x - x_j) \]

\[ + \frac{1}{3\sqrt{r^2 + c^2}} \left( r^2 + 2c^2 - \frac{c^3}{\sqrt{r^2 + c^2} + c} \right) \]  \hspace{1cm} (A4)

\[ \hat{u}_{j,yy} = \frac{1}{3(r^2 + c^2)^{3/2}} \left[ r^2 + c^2 - \frac{2\sqrt{r^2 + c^2} + c}{(\sqrt{r^2 + c^2} + c)^2} \right] (y - y_j) \]

\[ + \frac{1}{3\sqrt{r^2 + c^2}} \left( r^2 + 2c^2 - \frac{c^3}{\sqrt{r^2 + c^2} + c} \right) \]

\[ \hat{u}_{j,xy} = \frac{1}{3(r^2 + c^2)^{3/2}} \left[ r^2 + c^2 - \frac{2\sqrt{r^2 + c^2} + c}{(\sqrt{r^2 + c^2} + c)^2} \right] (x - x_j)(y - y_j) \]
It can be readily proved that
\[
\lim_{r \to 0} \hat{u}_{j,x} = 0, \quad \lim_{r \to 0} \hat{u}_{j,y} = 0
\]
\[
\lim_{r \to 0} \hat{u}_{j,xx} = \frac{c}{2}, \quad \lim_{r \to 0} \hat{u}_{j,yy} = \frac{c}{2}, \quad \lim_{r \to 0} \hat{u}_{j,xy} = 0
\] \hspace{1cm} (A5)

REFERENCES