Elastic flexural buckling analysis of composite beams of variable cross-section by BEM

E.J. Sapountzakis*, G.C. Tsiatas

School of Civil Engineering, National Technical University of Athens, Zografou Campus, GR - 157 80 Athens, Greece

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Abstract

In this paper a boundary element method is developed for the elastic flexural buckling analysis of composite Euler–Bernoulli beams of arbitrary variable cross-section. The composite beam consists of materials in contact. Each of these materials can surround a finite number of inclusions or openings. All of the cross-section’s materials are firmly bonded together. Since the cross-sectional properties of the beam vary along its axis, the coefficients of the governing differential equation are variable. The beam is subjected to a compressive centrally applied load together with arbitrarily axial and transverse distributed loading, while its edges are restrained by the most general linear boundary conditions. The resulting boundary value problems are solved using the analog equation method, a BEM based method. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the displacements as well as the stress resultants are computed at any cross-section of the beam using the respective integral representations as mathematical formulae. Several beams are analysed to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. The influence of the boundary conditions on the buckling load is demonstrated through examples with great practical interest. The flexural buckling analysis of a homogeneous beam is treated as a special case.

Keywords: Flexural buckling; Composite beam; Variable cross-section; Boundary integral equation; Analog equation method

1. Introduction

Elastic stability of beams is one of the most important criteria in the design of structures subjected to compressive loads. The flexural buckling coupled analysis is much more complicated in the general case of a composite beam of variable cross-section. Namely, a beam consisting of a relatively weak matrix material reinforced by stronger inclusions or of materials in contact. The extensive use of the aforementioned structural elements necessitates a reliable and accurate analysis of the flexural buckling problem.

Although there is extensive research on the coupled flexural buckling analysis of homogeneous beams of variable cross-section using analytical [1–5], semi-analytical [6] or numerical methods [7,8], to the authors’ knowledge, publications on the solution to the general composite problem do not exist.

In this investigation, an integral equation technique is developed for the flexural buckling analysis of composite beams of arbitrary variable cross section. The composite beam consists of materials in contact. Each of these materials can surround a finite number of inclusions or openings. All of the cross-section’s materials are firmly bonded together. The beam is subjected to a compressive centrally applied load together with arbitrarily axial and transverse distributed loading, while its edges are restrained by the most general linear boundary conditions. The solution method is based on the concept of the analog equation [9] as this is developed for the buckling analysis of beams [10]. According to this method, the fourth order ordinary differential equation with variable coefficients is replaced by an equivalent one pertaining to the transverse deformation of a substitute beam with unit bending stiffness subjected to a fictitious transverse load distribution under the same boundary conditions. Several beams are analysed to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. The influence of the boundary conditions on the buckling load is demonstrated through examples with great practical interest. The flexural buckling analysis of a homogeneous beam is treated as a special case.
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2. Statement of the problem

Let us consider an initially straight Euler–Bernoulli composite beam of length \( l \) having a doubly symmetric arbitrary variable cross-section. The cross-section variation is assumed to be smooth, so that the Euler beam theory is valid \([11]\). The beam is subjected to a compressive centrally applied load \( P \) and to the combined action of distributed loading \( p_x = p_x(x), p_y = p_y(x) \) and \( p_z = p_z(x) \) in the \( x, y \) and \( z \) directions, respectively, where the \( x \) axis coincides with the beam centroidal axis and \( y, z \) are the cross-section’s axes of symmetry (Fig. 1(a)). The composite cross-section consists of materials firmly bonded together, with modulus of elasticity \( E_j \), occupying the regions \( \Omega_j (j = 1, 2, \ldots, K) \) of the \( y, z \) plane. The materials of these regions are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions \( \Omega_j \) be denoted by \( \Gamma_j (j = 1, 2, \ldots, K) \). These boundary curves are piecewise smooth, i.e. they may have a finite number of corners (Fig. 1(b)).

Taking into account the two planes of symmetry of the beam cross-section, the flexural buckling problem can be uncoupled into two independent problems considering bending in both \( xz \) and \( yz \) planes. Thus, the evaluation of the buckling load will be accomplished by regarding the beam subjected separately to bending in \( xz \) and in \( yz \) planes and selecting the smaller buckling load obtained from these two cases.

Thus, considering the beam subjected to the compressive load \( P \) and to the combined action of distributed loading \( p_x = p_x(x) \), \( p_y = p_y(x) \), the angle of rotation of the cross-section according to the linear theory of beams satisfies the following relations

\[
\cos \theta \simeq 1 \quad (1a)
\]

\[
\sin \theta \simeq \frac{dw}{dx} \simeq \theta \quad (1b)
\]

while the curvature of the beam is given as

\[
\kappa = \frac{d\theta}{dx} \simeq \frac{d^2w}{dx^2}. \quad (2)
\]

Referring to Fig. 2, the stress resultants \( H, V_z \) acting in the \( x, z \) directions, respectively, are related to the axial \( N \) and the shear \( Q_z \) forces as

\[
H = N \cos \theta - Q_z \sin \theta \quad (3a)
\]

\[
V_z = N \sin \theta + Q_z \cos \theta \quad (3b)
\]

which by virtue of Eqs. (1a) and (1b) become

\[
H = N - Q_z \frac{dw}{dx} \quad (4a)
\]

\[
V_z = N \frac{dw}{dx} + Q_z. \quad (4b)
\]

The second term on the right hand side of Eq. (4a), expresses the influence of the shear force \( Q_z \) on the horizontal stress resultant \( H \). However, this term can be neglected since \( Q_z \) is much smaller than \( N \) and thus Eq. (4a) can be written as

\[
H \simeq N. \quad (5)
\]

The governing equation will be derived by considering the equilibrium of the deformed element. Thus, referring to Fig. 2 we obtain

\[
\frac{dH}{dx} = -p_x \quad (6a)
\]

\[
\frac{dV_z}{dx} = -p_z \quad (6b)
\]

\[
\frac{dM_y}{dx} = Q_z. \quad (6c)
\]
Substituting Eqs. (5) and (4b) into Eqs. (6a) and (6b) and using Eq. (6c) to eliminate \( Q \), we obtain

\[
\frac{dN}{dx} = -p_x, \quad \frac{d}{dx} \left( N \frac{dw}{dx} + \frac{dM_y}{dx} \right) = -p_z. \tag{7a}
\]

Eq. (7a) can be solved independently to evaluate the axial force \( N \). Thus, integrating the aforementioned equation we obtain

\[
N(x) = -\int_0^x p_x \, dx + c \tag{8}
\]

which by virtue of the boundary condition at the beam end \( x = l \)

\[
N(l) = -P
\]

yields for the unknown constant \( c \)

\[
c = -P + \int_0^l p_x \, dx. \tag{10}
\]

Substituting Eq. (10) into Eq. (8) yields

\[
N(x) = -P - \int_0^x p_x \, dx + \int_0^l p_x \, dx
\]

\[
= -P + \int_x^l p_x \, dx = -P + P_d(x). \tag{11}
\]

Using the above equation to eliminate the axial force \( N \) from Eq. (7b) we obtain

\[
\frac{d^2M_y}{dx^2} + \left( \frac{dp_d}{dx} \frac{dw}{dx} + P_d \frac{d^2w}{dx^2} \right) - \frac{P d^2w}{dx^2} = -p_z. \tag{12}
\]

The expression for the bending moment of a composite beam can be written as

\[
M_y = -D_y(x)\kappa = -D_y(x)\frac{d^2w}{dx^2}, \tag{13}
\]

where

\[
D_y(x) = \sum_{j=1}^K E_j I_{yj}(x) \tag{14}
\]

is its variable bending stiffness. Substituting Eq. (13) into Eq. (12) we obtain the governing differential equation of the problem as

\[
\frac{d^2}{dx^2} \left[ D_y(x) \frac{d^2w}{dx^2} \right] - \left( \frac{dp_d}{dx} \frac{dw}{dx} + P_d \frac{d^2w}{dx^2} \right) + P \frac{d^2w}{dx^2} = p_z
\]

inside the beam. \( \tag{15} \)

Moreover, the pertinent boundary conditions of the problem are given as

\[
\alpha_1 w(x) + \alpha_2 V_x(x) = \alpha_3 \tag{16a}
\]

\[
\beta_1 \frac{dw}{dx} + \beta_2 M_y(x) = \beta_3 \quad \text{at the beam ends } x = 0, l. \tag{16b}
\]

where \( \alpha_i, \beta_i \ (i = 1, 2, 3) \) are given constants. Eqs. (16a) and (16b) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately the functions \( \alpha_i \) and \( \beta_i \) (e.g. for a clamped edge it is \( \alpha_1 = \beta_1 = 1 \), \( \alpha_2 = \alpha_3 = 0 \), \( \beta_2 = \beta_3 = 0 \)).

Similarly, considering the beam subjected to the compressive load \( P \) and to the combined action of distributed loading \( p_x = p_x(x), p_z = p_z(x) \) we obtain the boundary value problem of the flexural buckling problem concerning bending in \( xy \) plane as

\[
\frac{d^2}{dx^2} \left[ D_y(x) \frac{d^2w}{dx^2} \right] - \left( \frac{dp_d}{dx} \frac{dw}{dx} + P_d \frac{d^2w}{dx^2} \right) + P \frac{d^2w}{dx^2} = p_y
\]

inside the beam \( \tag{17} \)

\[
\gamma_1 v(x) + \gamma_2 V_y(x) = \gamma_3
\]

\[
\delta_1 \frac{dv}{dx}(x) + \delta_2 M_x(x) = \delta_3 \quad \text{at the beam ends } x = 0, l. \tag{18b}
\]

where \( \gamma_i, \delta_i \ (i = 1, 2, 3) \) are given constants.

### 3. Integral representations — numerical solution

According to the preceding analysis, the flexural buckling problem of a beam reduces in establishing the displacement components \( w(x), V(x) \) having continuous derivatives up to the fourth order satisfying the governing Eqs. (15) and (17) inside the beam and the boundary conditions (16) and (18) at the beam ends \( x = 0, l \), respectively. The numerical solution of the boundary value problems described by Eqs. (15)–(18) is similar. For this reason, in the following we analyze the solution of the problem of Eqs. (15) and (16) noting any alteration or addition for the problem of Eqs. (17) and (18).

Eq. (15) is solved using the Analog Equation Method as it is developed for ordinary differential equations in [10,12]. This method is applied for the problem at hand as follows. Let \( w(x) \) be the sought solution of the boundary value problem described by Eqs. (15), (16a) and (16b). Differentiating this function four times yields

\[
\frac{d^4w}{dx^4} = b(x). \tag{19}
\]

Eq. (19) indicates that the solution of Eq. (15) can be established by solving Eq. (19) under the same boundary conditions (16a) and (16b), provided that the fictitious load distribution \( b(x) \) is first established. This distribution can be obtained using BEM.

Following the numerical procedure analytically presented in [10,12] and employing the constant element assumption, the discretized integral form of the solution of Eq. (19) and its derivatives at the \( N \) collocation points (see Fig. 3) are

\[
w = C_4 b - \left( E_1 \dot{w} + E_2 \ddot{w}_x + E_3 \dddot{w}_{xx} + E_4 \ddddot{w}_{xxx} \right), \tag{20a}
\]

\[
w_x = C_3 b - \left( E_1 \dot{w}_x + E_2 \ddot{w}_{xx} + E_3 \dddot{w}_{xxx} \right), \tag{20b}
\]
where \( C_i \) \((i = 1, 2, 3, 4)\) are \( N \times N \) known matrices; \( E_i \) \((i = 1, 2, 3, 4)\) are \( N \times 2 \) also known matrices and \( w, w_x, w_{xxx}, w_{xxxx} \) are vectors including the values of the transverse deflection \( w(x) \) and its derivatives at the \( N \) nodal points.

Moreover,

\[
\begin{align*}
\dot{w} &= (w(0) \ w(l))^T, \\
\dot{w}_x &= \left\{ w_x \right\}_{x=0}^{x=l} \\
\dot{w}_{xx} &= \left\{ \frac{dw(x)}{dx} \right\}_{x=0}^{x=l} \\
\dot{w}_{xxx} &= \left\{ \frac{d^2 w(x)}{dx^2} \right\}_{x=0}^{x=l} \\
\dot{w}_{xxxx} &= \left\{ \frac{d^3 w(x)}{dx^3} \right\}_{x=0}^{x=l}
\end{align*}
\]

are vectors including the two unknown boundary values of the respective boundary quantities and \( b = \{b_1 b_2 \ldots b_N\}^T \) is the vector including the \( N \) unknown nodal values of the fictitious load.

Employing the aforementioned procedure for the boundary conditions (16a) and (16b), the following set of linear equations is obtained

\[
\begin{bmatrix}
E_{11} & E_{12} & 0 & E_{14} \\
0 & E_{22} & E_{23} & 0 \\
E_{31} & E_{32} & E_{33} & E_{34} \\
0 & E_{42} & E_{43} & E_{44}
\end{bmatrix}
\begin{bmatrix}
\dot{w} \\
\dot{w}_x \\
\dot{w}_{xx} \\
\dot{w}_{xxx}
\end{bmatrix}
= \begin{bmatrix}
\alpha_3 \\
\beta_3 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
F_3 \\
F_4 \\
F_4
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\tag{22}
\]

where \( E_{ii}, (i = 1, 2, 4) \) are \( 2 \times 2 \) square matrices including the values of the functions \( a_1, a_2, \beta_1, \beta_2 \) of Eqs. (16a) and (16b); \( E_{ij}, (i = 3, 4, j = 1, 2, 3, 4) \) are square \( 2 \times 2 \) known coefficient matrices resulting from the values of the kernels at the beam ends; \( \alpha_3, \beta_3 \) are \( 2 \times 1 \) known column matrices including the boundary values of the functions \( a_3, \beta_3 \) of Eqs. (16a) and (16b) and \( F_j \) \((i = 3, 4)\) are \( 2 \times N \) rectangular known matrices originating from the integration of the kernels on the axis of the beam.

Eqs. (20) after eliminating the boundary quantities employing Eqs. (22), can be written as

\[
\begin{align*}
w &= Tb + t, \\
w_x &= T_x b + t_x, \\
w_{xx} &= T_{xx} b + t_{xx}, \\
w_{xxx} &= T_{xxx} b + t_{xxx}, \\
w_{xxxx} &= b
\end{align*}
\tag{23}
\]

where \( T, T_x, T_{xx}, T_{xxx} \) are known \( N \times N \) matrices and \( t, t_x, t_{xx}, t_{xxx} \) are known \( N \times 1 \) matrices. It is worth noting here that for homogeneous boundary conditions \((\alpha_3 = \beta_3 = 0)\) it is

\[
t = t_x = t_{xx} = t_{xxx} = 0.
\]

In the conventional BEM, the load vector \( b \) is known and Eqs. (23) are used to evaluate the transverse deflection \( w \) and its derivatives at the \( N \) nodal points. This, however, cannot be done here since \( b \) is unknown. For this purpose, \( N \) additional equations are derived, which permit the establishment of \( b \).

These equations result by applying Eq. (15) to the \( N \) collocation points, leading to the formulation of the following set of \( N \) simultaneous equations

\[
(A + PT_{xx}) b = c.
\tag{24}
\]

In the above equation the matrices \( A \) and \( c \) are evaluated from the expressions

\[
A = D + 2D_x T_{xxx} + D_{xx} T_x + P_{x1} T_{xx} + P_{x2} T_x, \tag{25a}
\]

\[
c = p_c - (2D_x t_{xxx} + D_{xx} t_x + P_{x1} t_{xx} + P_{x2} t_x) \tag{25b}
\]

where \( D, D_x, D_{xx} \) are \( N \times N \) diagonal matrices including the values of the stiffness and its derivatives at the nodal points. The values of the derivatives of the bending stiffness are approximated by appropriate central, forward or backward finite differences. Moreover, \( P_{x1}, P_{x2} \) are also \( N \times N \) diagonal matrices including the values of \( P_{dx}, \frac{dP_{dx}}{dx} \) at the nodal points and \( p_c \) is the vector containing the values of the external transverse distributed loading at the nodal points.

Solving the linear system of Eqs. (24) for the fictitious load \( b \), the deflection and its derivatives in the interior of the beam are computed using Eqs. (23).

### 3.1. Buckling equation

In this case it is \( \alpha_3 = \beta_3 = 0 \) (homogeneous boundary conditions) and \( p_c = 0 \). Thus, Eq. (24) becomes

\[
(A + PT_{xx}) b = 0. \tag{26}
\]

The condition that Eq. (26) has a non-trivial solution yields the buckling equation

\[
\det(A + PT_{xx}) = 0. \tag{27}
\]

### 4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method.

**Example 1.** For comparison reasons, a beam of length \( l \) with three different types of boundary conditions at the beam ends and corresponding varying stiffnesses has been studied. The first one of these types has hinged–hinged boundary conditions and stiffness variation according to the relation

\[
D(x) = D_0 \left( 1 + \xi - \xi^2 \right).
\]
Table 1
Buckling load \( P \) of the beam of Example 1

<table>
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<tbody>
<tr>
<td>Hinged–hinged</td>
<td>12.000</td>
<td>12.000</td>
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<tr>
<td>Fixed–hinged</td>
<td>12.000</td>
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<tr>
<td>Fixed–fixed</td>
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Fig. 4. Longitudinal (a) and transverse (b) sections of the variable I-section beam of Example 2.

The second one of the aforementioned types has fixed–hinged boundary conditions and stiffness variation given from

\[
D(x) = D_0 \left( \frac{9}{16} + \frac{3}{4} \xi - \xi^2 \right)
\]

while the final one has fixed–fixed boundary conditions and stiffness variation according to the relation

\[
D(x) = D_0 \left( \frac{1}{6} + \xi - \xi^2 \right)
\]

where \( \xi = x/l \). In Table 1 the computed buckling load \( P = P/(D_0/l^2) \) for the aforementioned cases is presented as compared with those obtained from an analytical solution [5] (in all three cases the exact buckling load is \( 12D_0/l^2 \)). The accuracy of the obtained results using the proposed procedure is remarkable.

Example 2. The elastic stability of the composite I-section beam (flange material \( E_f = 2.1 \times 10^8 \) kN/m\(^2\), web material \( E_w = 7.0 \times 10^7 \) kN/m\(^2\)) of Fig. 4, with length \( l = 1.0 \) m, has been studied. The flange width \( b_f(x) = b_0 - r_1x \) and the web height \( h_w(x) = h_0 - r_2x \) are both varying linearly, while all other dimensions remain constant; \( r_1, r_2 \) are constants stating the rate of change of the cross-section. Since Boley [11] has shown that the discrepancy between 2D elasticity and engineering beam theory for a beam with unit constant width and a rate of change of the cross-section \( r \approx 0.35 \) leads to an error of 7.5%, while for \( r \approx 0.17 \) the error is 1.8%, the maximum value of the aforementioned constant \( r_2 \) has been set equal to 0.16, for the studied example. Three types of boundary conditions are considered. In Figs. 5–7 the variation of the
buckling load $P$ with the rate of change $r_2$ for various values of $r_1$ for the cases of hinged–hinged, fixed–hinged and fixed–fixed boundary conditions, respectively are presented concerning bending in $xz$ plane. The influence of the boundary conditions on the buckling load is remarkable. Moreover, concerning bending in $yz$ plane, the computed values of the buckling load for (i) $r_1 = 0, r_2 = 0$, (ii) $r_1 = 0.02, r_2 = 0.0$ and (iii) $r_1 = 0.04, r_2 = 0.0$ and for various cases of boundary conditions are presented in Table 2 as compared with those obtained from an analytic solution and a FEM one [13] using (i) 14,000, (ii) 13,490 and (iii) 12,720 shell elements, respectively.

From this table the increased accuracy of the proposed method compared with the FEM one is easily verified. It is worth noting here the major merit of the aforementioned accuracy of the proposed method compared with 3D FEM solutions, arising from the disadvantages of the latter due to the difficulties of

- support modelling,
- discretization of the structure including thin walled members (shear-locking, membrane-locking [14]),
- increased number of degrees of freedom leading to severe or unrealistic computational time.

Moreover, the use of shell elements cannot give accurate results since the warping of the walls of a cross-section cannot be taken into account (midline model).

**Example 3.** The elastic stability of the composite beam of Fig. 8 with length $l = 5.0$ m, consisting of a steel I-beam ($E_s = 2.1 \times 10^8$ kN/m$^2$) of variable flange width filled with concrete ($E_c = 3.0 \times 10^7$ kN/m$^2$) has been studied. The flange width varies according to the law $b_f(x) = b_1 - 2a \sin(x\pi/l)$ while all other dimensions remain constant; $b_1 = 0.60$ m and $a$ is a shape parameter of the cross-section. In order to demonstrate the applicability and efficiency of the method the beam is subjected to a constant tensile distributed loading $p_x$, which gives $P_d = p_x(l - x)$ and $P_{d,x} = -p_x$. The computed values of the buckling load $P$ for both $p_x = 0$ and $p_x = 5000$ kN/m for various values of the shape parameter $a$ and for the cases of hinged–hinged, fixed–hinged and fixed–fixed boundary conditions are presented in Table 3. As it was expected the buckling load is decreased in the presence of the tensile distributed loading $p_x$. Moreover, the influence of the boundary conditions on the buckling load is once more verified.

**5. Concluding remarks**

In this paper a boundary element method is developed for the flexural buckling analysis of composite Euler–Bernoulli...
beams of arbitrary variable cross-section. The composite beam consists of materials in contact. Each of these materials can surround a finite number of inclusions or openings. All of the cross-section’s materials are firmly bonded together. The beam is subjected to a compressive centrally applied load together with arbitrarily axial and transverse distributed loading, while its edges are restrained by the most general linear boundary conditions. The main conclusions that can be drawn from this investigation are

(a) The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary variable cross-section, subjected to any linear boundary conditions and to an arbitrarily distributed or concentrated loading.

(b) Accurate results are obtained using a relatively small number of beam elements.

(c) The displacements as well as the stress resultants are computed at any cross-section of the beam using the respective integral representations as mathematical formulæ.

(d) The significant influence of the boundary conditions on the buckling load is remarkable.

(e) The flexural buckling analysis of a homogeneous beam can be treated as a special case.

(f) The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

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