Monoidal Grothendieck Construction

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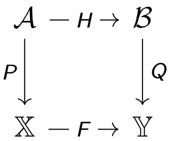
Outline

1. Fibrations and indexed categories
2. Monoidal Grothendieck Construction
3. Global and fibrewise monoidal structures
4. Examples

Fibrations and Indexed Categories

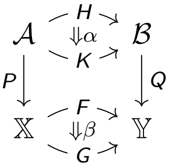
- There is a 2-category **Fib** of fibrations $P\colon \mathcal{A} \to \mathbb{X}$,

fibred 1-cells



where H is cartesian

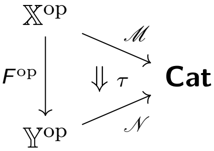
fibred 2-cells



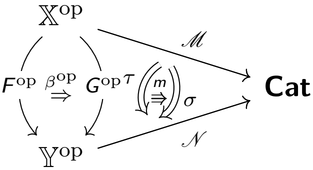
with $Q\alpha = \beta_P$

- There is a 2-category **ICat** of pseudofunctors $\mathscr{M}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$,

indexed 1-cells (F, τ)



indexed 2-cells (β, m)



The Grothendieck construction

There exists a 2-equivalence $\mathbf{Fib} \simeq \mathbf{ICat}$.

Given $\mathscr{M} \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$, the Grothendieck category $\int \mathscr{M}$ has

- objects (x, a) where $x \in \mathbb{X}$, $a \in \mathscr{M}x$
- morphisms $(x, a) \to (y, b)$ are $f \colon x \to y$ in $\mathbb{X}$, $a \to (\mathscr{M}f)(b)$ in $\mathscr{M}x$

The fibration $\int \mathscr{M} \to \mathbb{X}$ projects to the $\mathbb{X}$-parts.

Both 2-categories are cartesian monoidal:

$(\mathbf{Fib}, \times, 1_{\mathbf{1}})$ $\mathcal{A} \times \mathcal{B} \xrightarrow{P \times Q} \mathbb{X} \times \mathbb{Y}$ is a fibration when P, Q are

$(\mathbf{ICat}, \otimes, \Delta\mathbf{1})$ $\mathbb{X}^{\mathrm{op}} \times \mathbb{Y}^{\mathrm{op}} \xrightarrow{\mathscr{M} \times \mathscr{N}} \mathbf{Cat} \times \mathbf{Cat} \xrightarrow{\times} \mathbf{Cat}$ is $\mathbb{X} \times \mathbb{Y}$-indexed

The above lifts to a (cartesian) monoidal 2-equivalence $\mathbf{Fib} \simeq \mathbf{ICat}$.

2-categories of pseudomonoids

► For $(\mathcal{K}, \otimes, I)$ monoidal 2-category, $\mathrm{PsMon}(\mathcal{K})$ is the 2-category of pseudomonoids, strong morphisms and 2-cells.

$$\boxed{\mathrm{PsMon}(\mathbf{Fib}) \equiv \mathbf{MonFib}}$$

- *Monoidal* fibration: monoidal base $\mathbb{W}$ and total $\mathcal{V}$, strict monoidal $\mathcal{V} \xrightarrow{T} \mathbb{W}$, cartesian $\otimes_{\mathcal{V}}$

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes_{\mathcal{V}}} & \mathcal{V} \\ {\scriptstyle T\times T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathbb{W} \times \mathbb{W} & \xrightarrow{\otimes_{\mathbb{W}}} & \mathbb{W} \end{array}$$

- *Monoidal* fibred 1-cell is (H, F) both monoidal functors
- *Monoidal* fibred 2-cell is (α, β) both monoidal natural

$$\boxed{\mathrm{PsMon}(\mathbf{ICat}) \equiv \mathbf{MonICat}}$$

- *Monoidal* indexed category: monoidal domain $\mathbb{W}$, lax monoidal pseudofunctor $\mathbb{W}^{\mathrm{op}} \xrightarrow{\mathscr{M}} \mathbf{Cat}$

$$\phi_{x,y} \colon \mathscr{M}x \times \mathscr{M}y \to \mathscr{M}(x \otimes_{\mathbb{W}} y)$$

$$\text{for all } x, y \in \mathbb{W}$$

$$\phi_0 \colon \mathbf{1} \to \mathscr{M}(I_{\mathbb{W}})$$

- *Monoidal* indexed 1-cell (F, τ), F monoidal, τ monoidal pseudonatural
- *Monoidal* indexed 2-cell (β, m), β mon natural, m mon modification

Monoidal Grothendieck Construction

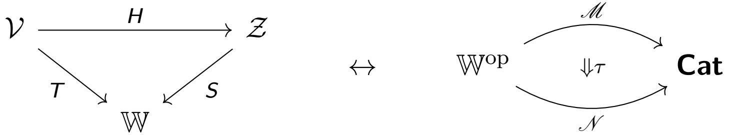
There is a 2-equivalence $\mathbf{MonFib} \simeq \mathbf{MonICat}$.

For lax monoidal pseudofunctor $(\mathscr{M}, \phi, \phi_0)\colon \mathbb{W}^{\mathrm{op}} \to \mathbf{Cat}$, equip $\int \mathscr{M}$ with

$$(x, a) \otimes (y, b) := (x \otimes_{\mathbb{W}} y, \phi_{x,y}(a, b)), \quad I := (I_{\mathbb{W}}, \phi_0(*))$$

⋆ Lax monoidal structure gives a 'global' tensor product to $\int \mathscr{M} \to \mathbb{W}$.

Fixing the monoidal base, there is a 2-equivalence
$\mathbf{MonFib}(\mathbb{W}) \simeq \mathbf{Mon2Cat}_{ps}(\mathbb{W}^{\mathrm{op}}, \mathbf{Cat})$.



Fibrewise monoidal structure

Start over, from $\mathbf{Fib}(\mathbb{X}) \simeq \mathbf{ICat}(\mathbb{X})$.

These 2-categories also monoidal:

$(\mathbf{Fib}(\mathbb{X}), \times_{\mathbb{X}}, 1_{\mathbb{X}})$ $\quad$ $\mathcal{A} \times_{\mathbb{X}} \mathcal{B} \to \mathbb{X}$ pullback of fibrations

$(\mathbf{ICat}(\mathbb{X}), \otimes, \Delta\mathbf{1})$ $\quad$ $\mathbb{X}^{\mathrm{op}} \to \mathbb{X}^{\mathrm{op}} \times \mathbb{X}^{\mathrm{op}} \xrightarrow{\mathscr{M} \times \mathscr{N}} \mathbf{Cat} \times \mathbf{Cat} \xrightarrow{\times} \mathbf{Cat}$

▶ Pseudomonoid in $\mathbf{Fib}(\mathbb{X})$? Ordinary fibration whose fibres are monoidal, reindexing functors are strong monoidal.

▶ Pseudomonoid in $\mathbf{ICat}(\mathbb{X})$? Pseudofunctor $\mathbb{X}^{\mathrm{op}} \to \mathbf{MonCat}$.

There is a 2-equivalence $\mathrm{PsMon}(\mathbf{Fib}(\mathbb{X})) \simeq \mathbf{2Cat}_{\mathsf{ps}}(\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat})$.

$\star$ $\int \mathscr{M} \to \mathbb{X}$ obtains 'fibrewise' monoidal structure; in general, this does not give a 'global' one! $\mathbb{X}$ is an arbitrary category.

Cartesian base $\mathbb{X}$

If $\mathbb{X}$ is cartesian monoidal, all above structures are equivalent

$$\begin{array}{ccc}
\mathbf{MonFib}(\mathbb{X}) & \xrightarrow{\simeq} & \mathbf{Mon2Cat}_{ps}(\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}) \\
\downarrow \wr & \overset{(*)}{\searrow} & (**)\ \downarrow \wr \\
\mathrm{PsMon}(\mathbf{Fib}(\mathbb{X})) & \xrightarrow{\simeq} & \mathbf{2Cat}_{ps}(\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat})
\end{array}$$

- Shulman constructs $(*)$ in 'Framed bicategories, monoidal fibrations'
- Can obtain $(**)$ via equivalences involving $\mathbf{Mon2Cat}_{ps}$ and $\mathbf{2Cat}_{ps}$.

When $\mathbb{X}$ is cartesian, 'monoidalness' transfers from the target category to the structure of the functor and vice versa.

Global categories of modules and comodules

Suppose $(\mathcal{V}, \otimes, I, \sigma)$ is braided monoidal.

- Categories of monoids $\mathbf{Mon}(\mathcal{V})$, comonoids $\mathbf{Comon}(\mathcal{V})$ are monoidal.

$$
\begin{array}{ccc}
\mathbf{Mon}(\mathcal{V})^{\mathrm{op}} & \longrightarrow & \mathbf{Cat} \\
A & \mapsto & \mathbf{Mod}_{\mathcal{V}}(A) \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^*} \\
B & \mapsto & \mathbf{Mod}_{\mathcal{V}}(B)
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{Comon}(\mathcal{V}) & \longrightarrow & \mathbf{Cat} \\
C & \mapsto & \mathbf{Comod}_{\mathcal{V}}(C) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_!} \\
D & \mapsto & \mathbf{Comod}_{\mathcal{V}}(D)
\end{array}
$$

are lax monoidal: for M an A-module, N a B-module, $M \otimes N$ is $A \otimes B$-module via $A \otimes B \otimes M \otimes N \xrightarrow{\sim} A \otimes M \otimes B \otimes N \xrightarrow{\mu \otimes \mu} M \otimes N$.

- They give rise to (split) monoidal (op)fibrations

$$\mathbf{Mod} \to \mathbf{Mon}(\mathcal{V}) \qquad \mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$$

⋆ These do not fall under the fibrewise monoidal case.

Zunino and Turaev categories

▶Family fibration $\mathrm{Fam}(\mathcal{C})$ induced by the functor $[-,\mathcal{C}]\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$. $[X,\mathcal{C}]$ has $\{M_x\}_{x\in X}$ of $\mathcal{C}$-objects, $f\colon X \to Y$ induces reindexing f^*.

For $\mathcal{V}$ monoidal, $[-,\mathcal{V}]\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ is lax monoidal; gives rise to

- (split) monoidal fibration $\mathrm{Fam}(\mathcal{V}) \to \mathbf{Set}$, morphisms look like

$$\begin{cases} t\colon M_x \to N_{f(x)} \text{ in } \mathcal{V} \\ f\colon X \to Y \text{ in } \mathbf{Set} \end{cases}$$

- $(M \otimes N)_{X\times Y} = \{M_x \otimes_{\mathcal{V}} N_y\}_{x\in X, y\in Y}$
- (split) monoidal opfibration $\mathrm{Maf}(\mathcal{V}) \to \mathbf{Set}^{\mathrm{op}}$, morphisms look like

$$\begin{cases} s\colon M_{g(y)} \to N_y \text{ in } \mathcal{V} \\ g\colon Y \to X \text{ in } \mathbf{Set} \end{cases}$$

▶Caenepeel&De Lombaerde use the Zunino=:$\mathrm{Fam}(\mathbf{Mod}_\mathrm{R})$ and the Turaev=:$\mathrm{Maf}(\mathbf{Mod}_\mathrm{R})$ category to study Hopf group-(co)algebras.

⋆ Since $\mathbf{Set}$ is cartesian, these are both fibrewise monoidal as well.

Graphs and cospans

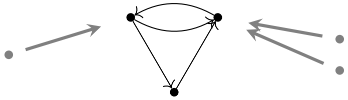
The functor $F\colon \mathbf{Set} \to \mathbf{Cat}$ which maps any set X to $E \underset{t}{\overset{s}{\rightrightarrows}} X$, the category of all graphs with vertices X, induces opfibration $\mathbf{Grph} \to \mathbf{Set}$.

- It has a lax monoidal structure $(\mathbf{Set}, +, 0) \to (\mathbf{Cat}, \times, \mathbf{1})$

$$\phi_{X,Y}\left(E \underset{t}{\overset{s}{\rightrightarrows}} X \,,\; D \underset{t}{\overset{s}{\rightrightarrows}} Y \right) = \; E + D \underset{t+t}{\overset{s+s}{\rightrightarrows}} X + Y$$

which induces cocartesian mon opfibration $(\mathbf{Grph}, +, 0) \to (\mathbf{Set}, +, 0)$.

▶Fong uses $\widetilde{F}\colon \mathbf{FinSet} \to \mathbf{Set}$ to *decorate* apices of cospans with graphs. Baez&Courser use monoidal $\int F \to \mathbf{Set}$ (+adjoint) to *structure* cospans.

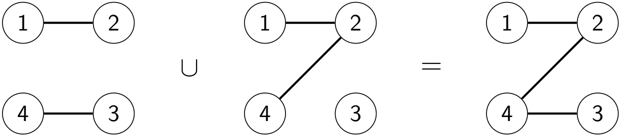


⋆ Not only base $\mathbf{Set}$, but also total category is cocartesian (fibres too).

Network Models

[Baez, Foley, Moeller, Pollard] Let $S(X)$ be the free symmetric monoidal category on a finite set X, e.g. $S(1) = \mathbf{FinSet}^{bij}$.

- *Network model* = symmetric lax monoidal $(S(X), \otimes, I) \to (\mathbf{Mon}, \times, 1)$.



▶It always induces a monoidal (split) opfibration: the underlying operad of the total category has algebras that model various networks.

Examples include simple/directed graphs, (directed) multigraphs, hypergraphs, graphs with colored edges/vertices, petri nets.

$\star$ Base $S(X)$ is not cocartesian; in many examples, it takes $+$ from **Set**.

Thank you for your attention!