Enrichment of $\mathcal{V}$-categories in $\mathcal{V}$-cocategories

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1) The bicategory of $\mathcal{V}$-matrices

$\mathcal{V}$ is a cocomplete symmetric monoidal closed category.

The bicategory $\mathcal{V}$-**Mat** of $\mathcal{V}$-matrices:

$\rightarrow$ objects are sets X, Y

$\rightarrow$ morphisms $X \xrightarrow{S} Y$ are functors $S : Y \times X \to \mathcal{V}$

$\rightarrow$ 2-cells $X \underset{S'}{\overset{S}{\Downarrow \sigma}} Y$ are natural transformations

The composite $X \xrightarrow{S} Y \xrightarrow{T} Z$ is given by

$$(S \circ T)(z, x) = \sum_{y \in Y} T(z, y) \otimes S(y, x).$$

- $\mathcal{V}$-**Mat**(X, X) is cocomplete, monoidal with $\circ$ (which commutes with colimits) and locally presentable when $\mathcal{V}$ is.

The category of $\mathcal{V}$-graphs

A function $X \xrightarrow{f} Y$ determines $X \overset{f_*}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!|\;\;\;\;\;\;} Y$, $Y \overset{f^*}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!|\;\;\;\;\;\;} X$ with

$$f_*(y,x) = f^*(x,y) = \begin{cases} I & \text{if } f(x)=y \\ 0 & \text{otherwise} \end{cases} \quad \text{and } f_* \dashv f^* \text{ in } \mathcal{V}\text{-}\mathbf{Mat}.$$

The category $\mathcal{V}$-**Grph** has objects $G \in \mathcal{V}\text{-}\mathbf{Mat}(X,X)$ and arrows

$$G_X \xrightarrow{F} H_Y \text{ pairs } \begin{cases} \bar{F} : G \Rightarrow f^* H f_* & \text{in } \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\ f : X \to Y & \text{in } \mathbf{Set} \end{cases}.$$

- $\mathcal{V}$-**Grph** is cocomplete, symmetric monoidal closed with

$$\mathrm{Hom}(G_X, H_Y)(k,l) := \prod_{x,x' \in X} [G(x,x'), H(kx, lx')]$$

for any $k, l \in Y^X$, and locally presentable when $\mathcal{V}$ is.

2) $\mathcal{V}$-categories and $\mathcal{V}$-cocategories

- A (small) $\mathcal{V}$-category is a monad in $\mathcal{V}$-**Mat**, i.e.

$$\mathcal{A}_X \in \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$$

and a $\mathcal{V}$-functor $F : \mathcal{A}_X \to \mathcal{B}_Y$ is a pair

$$\begin{cases} \bar{F} : A \Rightarrow f^* B f_* & \text{in } \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X)) \\ f : X \to Y & \text{in } \mathbf{Set} \end{cases}$$

- A (small) $\mathcal{V}$-*cocategory* is a comonad in $\mathcal{V}$-**Mat**, i.e.

$$\mathcal{C}_X \in \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X,X))$$

and a $\mathcal{V}$-*cofunctor* $F : \mathcal{C}_X \to \mathcal{D}_Y$ is a pair

$$\begin{cases} \hat{F} : f_* C f^* \Rightarrow D & \text{in } \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(Y,Y)) \\ f : X \to Y & \text{in } \mathbf{Set} \end{cases}$$

Proposition

1) $\mathcal{V}$-**Cat** is a symmetric monoidal category, monadic over $\mathcal{V}$-**Grph** and locally presentable when $\mathcal{V}$ is.
2) $\mathcal{V}$-**Cocat** is a symmetric monoidal category, cocomplete and the forgetful functor to $\mathcal{V}$-**Grph** has a right adjoint.

The internal hom in $\mathcal{V}$-**Grph** induces a functor

$$\mathrm{Hom} : \mathcal{V}\text{-}\mathbf{Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}.$$

Concretely: for a $\mathcal{V}$-cocategory $\mathcal{C} \in \mathbf{Comon}(\mathcal{V}\text{-}\mathbf{Mat}(X, X))$ and a $\mathcal{V}$-category $\mathcal{B} \in \mathbf{Mon}(\mathcal{V}\text{-}\mathbf{Mat}(Y, Y))$, the $\mathcal{V}$-graph

$$Y^X \xrightarrow{\mathrm{Hom}(\mathcal{C}_X, \mathcal{B}_Y)} Y^X$$

obtains the structure of a $\mathcal{V}$-category.

3) Enrichment of $\mathcal{V}$-Cat in $\mathcal{V}$-Cocat

The forgetful functor $\mathcal{V}$-**Grph** $\to$ **Set** is a bifibration.

Arises as the Grothendieck category for the pseudofunctors

$$\mathscr{M} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Cat} \qquad \& \qquad \mathscr{L} : \mathbf{Set} \longrightarrow \mathbf{Cat}$$

$$\begin{array}{ccc} X & \mapsto & \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^* \circ\!-\!\circ f_*} \\ Y & \mapsto & \mathcal{V}\text{-}\mathbf{Mat}(Y,Y) \end{array} \qquad\qquad \begin{array}{ccc} X & \mapsto & \mathcal{V}\text{-}\mathbf{Mat}(X,X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_* \circ\!-\!\circ f^*} \\ Y & \mapsto & \mathcal{V}\text{-}\mathbf{Mat}(Y,Y) \end{array}$$

$\mathcal{V}$-**Cat** is fibred over **Set** and $\mathcal{V}$-**Cocat** is opfibred over **Set**.

The functors $\mathscr{M}f$ and $\mathscr{L}f$ restrict to the categories of monoids and comonoids respectively.

The pair $(\mathrm{Hom}(-,\mathcal{B}_Y)^{\mathrm{op}}, Y^{(-)^{\mathrm{op}}})$ is an opfibred 1-cell between the opfibrations $U : \mathcal{V}\text{-}\mathbf{Cocat} \to \mathbf{Set}$ and $V^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathbf{Set}^{\mathrm{op}}$.

Theorem

The functor $\mathrm{Hom}^{\mathrm{op}}$ between the total categories has a parametrised adjoint $Q : \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \times \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cocat}$ with $\mathrm{Hom}(-,\mathcal{B}_Y)^{\mathrm{op}} \dashv Q(-,\mathcal{B}_Y)$ for every $\mathcal{V}$-category $\mathcal{B}$.

Theorem (*Janelidze, Kelly*)

If $* : \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ is an action and each $- * A$ has a right adjoint $F(A,-)$, then we can enrich $\mathcal{A}$ in $\mathcal{V}$ with hom-object functor F.

- $\mathrm{Hom}^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cocat} \times \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}$ is an action.

The categories $\mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}}$ and $\mathcal{V}\text{-}\mathbf{Cat}$ are enriched in the symmetric monoidal category $\mathcal{V}\text{-}\mathbf{Cocat}$, with $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A},\mathcal{B}) = Q(\mathcal{A},\mathcal{B})$.

4) General Pattern

- Measuring coalgebras and comodules

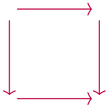
$$\begin{array}{ccc} \mathbf{Mod}^{\mathrm{op}} & \underset{\mathrm{Hom}(-,N_B)^{\mathrm{op}}}{\overset{Q(-,N_B)}{\rightleftarrows}} \top & \mathbf{Comod} \\ \downarrow & & \downarrow \\ \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} & \underset{H(-,B)^{\mathrm{op}}}{\overset{P(-,B)}{\rightleftarrows}} \top & \mathbf{Comon}(\mathcal{V}) \end{array}$$

- Enriched cocategories and categories

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} & \underset{\mathrm{Hom}(-,\mathcal{B}_Y)^{\mathrm{op}}}{\overset{Q(-,\mathcal{B}_Y)}{\rightleftarrows}} \top & \mathcal{V}\text{-}\mathbf{Cocat} \\ \downarrow & & \downarrow \\ \mathbf{Set}^{\mathrm{op}} & \underset{Y^{(-)\mathrm{op}}}{\overset{Y^{(-)}}{\rightleftarrows}} \top & \mathbf{Set} \end{array}$$

Enriched fibration $\rightarrow$ $\mathcal{V}$-(co)modules, $\mathcal{V}$-(co)operads

Thank you for your attention!



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