Abstract Dynamical Systems

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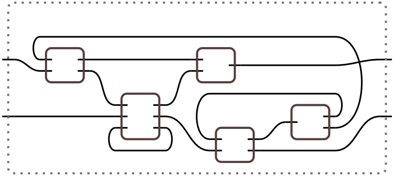
Goal: categorical framework for modeling and analysis of systems

systems as boxes
inhabitants



channels of info
flow as wires

Analyse the behavior of the composite system using analysis of the particular systems components and their wired interconnection.

- Coherent zoom in/out subsystems, due to *compositionality* [operad algebras]
- Appropriate notions of time for abstract systems [sheaves]

Outline

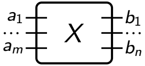
1. The operad of wiring diagrams
2. Interval sheaves
3. Continuous and discrete machines
4. Total and deterministic variations

Monoidal category of wiring diagrams

⋆ A $\mathcal{C}$*-typed finite set* is X together with typing function $X \xrightarrow{\tau} \mathrm{ob}\mathcal{C}$; these form a comma category $\mathbf{TFS}_{\mathcal{C}}$, cocartesian monoidal.

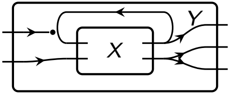
The monoidal category $\mathcal{W}_{\mathcal{C}}$ has

- objects *labeled boxes*, i.e. $X = (X^{\mathrm{in}}, X^{\mathrm{out}}) \in \mathbf{TFS}_{\mathcal{C}}^2$



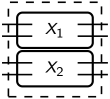
think of $X^{\mathrm{in/out}}$-elements as ports, their types as possible info values

- morphisms $(X^{\mathrm{in}} \xrightarrow{\phi^{\mathrm{in}}} X^{\mathrm{out}} + Y^{\mathrm{in}}, Y^{\mathrm{out}} \xrightarrow{\phi^{\mathrm{out}}} X^{\mathrm{out}}) \in \mathbf{TFS}_{\mathcal{C}}^2$



think of $\phi^{\mathrm{in/out}}$ expressing which port is fed info by which

- tensor product $X_1 \oplus X_2 = ({X_1}^{\mathrm{in}} + {X_2}^{\mathrm{in}}, {X_1}^{\mathrm{out}} + {X_2}^{\mathrm{out}})$



think of parallel placement of boxes

⋆ If $\mathcal{C}$ finitely complete, *dependent product* $\widehat{X} = \prod_x \tau(x)$ gives strong monoidal $\widehat{(-)}\colon \mathbf{TFS}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{C} \rightsquigarrow$ passage to $\mathcal{C}$-context.

Model systems as algebras for $\mathcal{W}_{\mathcal{C}} \Leftrightarrow$ the *underlying operad* $\mathcal{O}\mathcal{W}_{\mathcal{C}}$; monoidal world for formal language, operadic world for visual.

- A lax monoidal functor $F\colon \mathcal{W}_{\mathcal{C}} \to \mathbf{Cat}$ gives semantics to boxes, composite formula to wiring diagrams

$$F(X_1) \times \ldots \times F(X_n) \xrightarrow{F_{X_1 \ldots X_n}} F(X_1 + \ldots + X_n) \xrightarrow{F\phi} FY.$$

Dynamical Systems as Algebras

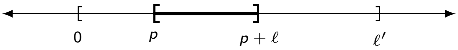
- Continuous (open) dynamical systems (previous talk)
- Discrete dynamical systems, or (finite case) Moore machines

Modeling Time: Categories of intervals

$\mathbb{R}_{\geq 0}$ positive reals, $\mathrm{Tr}_p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ translation-by-p.

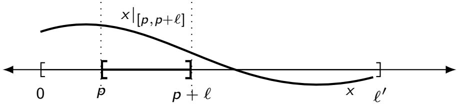
- Category **Int** of *continuous intervals* has objects $\mathbb{R}_{\geq 0}$, morphisms $\mathbf{Int}(\ell, \ell') = \{\mathrm{Tr}_p | p \in \mathbb{R}_{\geq 0} \text{ and } p \leq \ell' - \ell\}$; equivalently via image

$[0, \ell] \overset{p}{\subseteq} [0, \ell']$



- Category $\mathbf{Int}_N$ of *discrete intervals*, $\mathrm{ob} = \mathbb{N}$, $n \xrightarrow{\mathrm{Tr}_p} n'$ by $p \in \mathbb{N}$.

If A: $\mathbf{Int}^{\mathrm{op}} \to \mathbf{Set}$, view section $x \in A(\ell')$ & restriction $A(\mathrm{Tr}_p)(x)$



Sheaves on intervals

For $\ell \in \mathbf{Int}$ and $0 \leq p \leq \ell$, the pairs $p \xrightarrow{[0,p]} \ell$, $(\ell\text{-}p) \xrightarrow{[p,\ell]} \ell$ form a cover for ℓ. These generate a coverage for $\mathbf{Int}$; similarly for $\mathbf{Int}_N$.

$\star$ $\widetilde{\mathbf{Int}}$ and $\widetilde{\mathbf{Int}_N}$ are the toposes of *continuous* and *discrete interval sheaves*, i.e. $\mathbf{Int}_{(N)}$-presheaves whose compatible sections glue.

Examples

- $\widetilde{\mathbf{Int}_N} \simeq \mathbf{Grph}$, so every graph gives a discrete interval sheaf
- $\mathrm{F}\colon \mathbf{Set} \to \widetilde{\mathbf{Int}}$ by $\mathrm{F}(\mathrm{X})(\ell) = \{\mathrm{f}\colon [0,\ell] \to \mathrm{X}\}$, sheaf of functions
- $\mathrm{Ext}_\epsilon\colon \widetilde{\mathbf{Int}} \to \widetilde{\mathbf{Int}}$ by $\mathrm{Ext}_\epsilon(\mathrm{A})(\ell) = \mathrm{A}(\ell + \epsilon)$, ϵ-extension sheaf

Idea: $\widetilde{\mathbf{Int}_{(N)}}$-labeled boxes have ports carrying very general time-based signals, expressed as sheaves of ‘all possible behaviors’.

Abstract machines

- A *continuous machine* with input & output A & $B \in \widetilde{\textbf{Int}}$ is

$$A \xleftarrow{p^i} S \xrightarrow{p^o} B$$

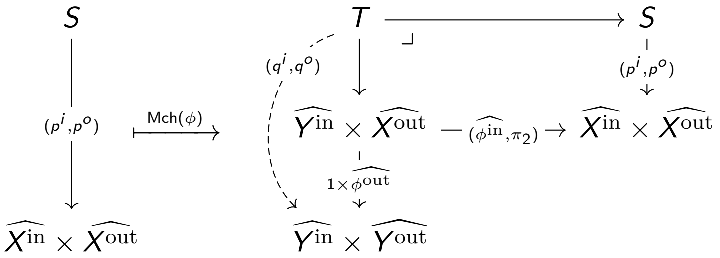
S - state sheaf
p^i - input sheaf map
p^o - output sheaf map

$\mathrm{Mch}(\mathrm{A}, \mathrm{B}) = \widetilde{\textbf{Int}}/_{\mathrm{A}\times\mathrm{B}}$ the topos of continuous (A, B)-machines.

- For $A, B \in \widetilde{\textbf{Int}}_N$, *discrete machines* $\mathrm{Mch}_\mathrm{N}(\mathrm{A}, \mathrm{B}) = \widetilde{\textbf{Int}}_\mathrm{N}/_{\mathrm{A}\times\mathrm{B}}$.

Continuous machines form a $\mathcal{W}_{\widetilde{\mathbf{Int}}}$-algebra

Functor $\mathsf{Mch}\colon \mathcal{W}_{\widetilde{\mathbf{Int}}} \to \mathbf{Cat}$ by $(X^{\text{in}}, X^{\text{out}}) \mapsto \mathsf{Mch}(\widehat{X^{\text{in}}}, \widehat{X^{\text{out}}})$ and



Finally, lax monoidal structure by taking products of spans:

$$\left(S \xrightarrow{(p^i,p^o)} \widehat{X^{\text{in}}} \times \widehat{X^{\text{out}}},\ T \xrightarrow{(q^i,q^o)} \widehat{Z^{\text{in}}} \times \widehat{Z^{\text{out}}}\right) \mapsto (p^i \times q^i, p^o \times q^o)$$

Total and deterministic machines

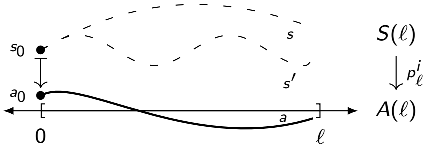
Characteristics of interest: for initial state and input, the machine

- uniquely evolves or 'stays idle' $\rightsquigarrow$ determinism
- always evolves $\rightsquigarrow$ totality

▶ Continuous machines 

are neither in general:

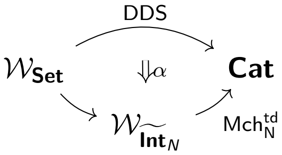
Starting in state germ s_0, for input a over ℓ-interval, there may or may not be s_0-extension



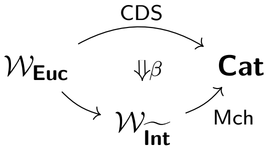
⋆ A *total* machine would have at least one extension , whereas a *deterministic* machine would have maximum one extension.

⋆ There exist subalgebras of $\mathsf{Mch}_{(\mathsf{N})} \colon \mathcal{W}_{\widetilde{\mathbf{Int}}} \to \mathbf{Cat}$ of total and deterministic machines, by imposing conditions on p^i and q^i.

- There are algebra maps from discrete dynamical systems



and from continuous dynamical systems



Algebra maps ‘translate’ between various processes; can then interconnect arbitrary systems & study them on common ground.

Thank you for your attention!