Welcome to Math 006B!

Modeling Relationships - the Bottle Problem

Idea: depending on the shape of a bottle, filling it water changes the height in some reasonable way.

- Cylinder: linear relationship, constant rate of change
- Conical: rate of change increases since width decreases
- Vase: rate of change increases or decreases according to the width

Average Rate of Change

The average rate of change for a function $y = f(x)$ from x_1 to x_2 is

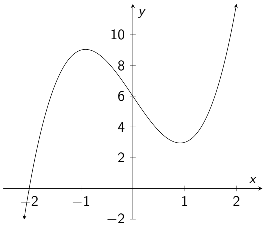
$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Concavity

When the average rate of change is increasing, the function is *concave up*; when it's decreasing, the function is *concave down*.
An *inflection point* is a point where concavity changes.

Work out the following: consider $f(x) = 2x^3 - 5x + 6$.

1. What is the averate rate of change, from $x = 1$ to $x = 2$? 9
2. What is the averate rate of change, from $x = -1$ to $x = 0$? -3
3. What is the domain of $f(x)$, in interval notation? $(-\infty, +\infty)$
4. Which are concave up and concave down intervals? $(0, +\infty)$, $(-\infty, 0)$



5. Which is the inflection point? $(0, 6)$

Polynomial Functions

Definition

A *polynomial* is an expression (standard form)

$$a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $n \in \mathbb{N}$ and $a_n, \ldots, a_0 \in \mathbb{R}$.

- the expressions $a_i x^i$ for some $i = 0, \ldots, n$ are the *terms*
- if $a_n \neq 0$, the *degree of the polynomial* is n (the highest exponent)
- $a_n x^n$ is the *leading term*, a_0 is the *constant term*
- $a_n, \ldots, a_0$ are the *coefficients*, a_n is the *leading* coefficient

Transformations

Transformations for graphs of polynomial functions $f(x)$

• *vertical shifts*: for $b > 0$, $y = f(x) \pm b$ shifts **up** or **down** b units

• *horizontal shifts*: for $d > 0$, $y = f(x \pm d)$ shifts **left** or **right** d units

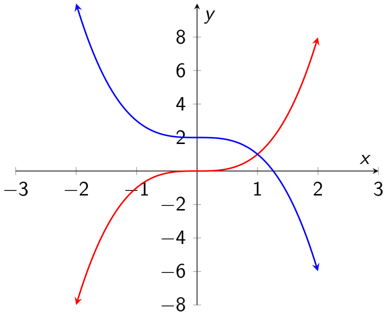
• *reflections*: $y = -f(x)$ with respect to x-axis, $y = f(-x)$ wrt y-axis

• *vertical stretch/compression*: $y = af(x)$ **stretches** if $a > 1$, **compresses** if $0 < a < 1$

• *horizontal stretch/compression*: $y = f(cx)$ **compresses** if $c > 1$, **stretches** if $0 < c < 1$

Work out the following:

1. which is the degree, leading coefficient and constant term of $x^2 - 8 + \frac{2}{3}x^5 - 2x$? $5, \frac{2}{3}, -8$
2. Write a function $f(x)$ of the shape of $y = x^2$, but shifted right by 6 units and upside-down. $f(x) = -(x-6)^2$
3. Using which shifts can we graph $y = (-x)^3 - 2$ from $y = x^3$? reflect with respect to y-axis, shift down by 2
4. Express $g(x)$ in terms of $f(x)$: $g(x) = 2 - f(x)$ or $g(x) = 2 + f(-x)$



Quadratic Functions

Idea: study polynomials of the form $\boxed{f(x) = ax^2 + bx + c}$ for $a, b, c \in \mathbb{R}$.

Zeros or Roots of functions

A *zero* or *root* of $f(x) = ax^2 + bx + c$ is a solution to the equation

$$f(x) = 0.$$

- Find roots by factoring, $f(x) = a(x - x_1)(x - x_2)$, and then using *zero product principle*: if $ab = 0$, then $a = 0$ or $b = 0$.

Quadratic Formula

$\Delta = b^2 - 4ac$ discriminant

The roots of any quadratic function $f(x) = ax^2 + bx + c$ are

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

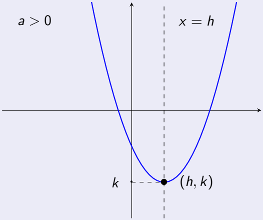
$\Delta > 0$: 2 real roots
$\Delta = 0$: 1 real roots
$\Delta < 0$: 2 complex roots

Graphs of quadratic functions: parabolas

Vertex form $f(x) = a(x-h)^2 + k$

The graph of $f(x)$ is a parabola that

- opens up if $a > 0$, down if $a < 0$
- (h, k) is its *vertex*
- $x = h$ is its axis of symmetry
- k is minimum if $a > 0$, max if $a < 0$



Graph of general $f(x) = ax^2 + bx + c$

The graph of $f(x)$ is a parabola that opens up or down ($a \lessgtr 0$) with vertex

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Also $f(-\frac{b}{2a})$ is the maximum or minimum value (based on  or ).

Work out the following:

1. Find the roots of $f(x) = (2x+5)(3x-1)$. $-\frac{5}{2}, \frac{1}{3}$
2. Find the roots of $k(x) = x^2 - 3x - 10$. 5, -2
3. Find the roots of $g(x) = x^2 + 9$ using the quadratic formula. $3i$, $-3i$
4. What is the vertex of $h(x) = -x^2 + 6x - 5$? $(3, 4)$
5. Does $h(x)$ it have a maximum or minimum? What is its value? max, 4

Roots and Multiplicities of Polynomial Functions

A polynomial function of degree n

$$f(x) = a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_2 x^2 + a_1 x + a_0$$

is called *linear* if $n = 1$, *quadratic* if $n = 2$, *cubic* if $n = 3$, *quartic* if $n = 4$.

- Like for quadratics, the zeros/roots of any function $f(x)$ are the solutions of $f(x) = 0$: *the inputs x_i whose output is $f(x_i) = 0$!*

For its graph $y = f(x)$, the x-intercepts are exactly the points $(x_i, 0)$.

- The *multiplicity* of a zero x_i is 'how many times' it occurs as a zero, i.e. how many times the factor $(x - x_i)$ repeats in the factored form of $f(x)$.

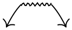
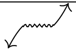
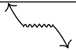
Every polynomial of degree $n \geq 1$ has at least 1 zero and at most n zeros.

SIGN TABLE use zeros of $f(x)$ to divide real line into intervals; the sign of their output determine the sign of ALL outputs in that interval.

Graphs of higher polynomials

Idea: the *monomial* leading term $a_n x^n$ determines far left and right!

End behavior: leading coefficient a_n & degree n

	$a_n > 0$	$a_n < 0$
n even		
n odd		

Methodology for graphing $p(x) = a_n x^n + \ldots + a_1 x + a_0$

- determine end behavior
- factor $p(x)$; find zeros ($p(x_i) = 0$) and x-intercepts $(x_i, 0)$
- sign table
- odd multiplicity $\rightsquigarrow$ graph crosses x-axis, even $\rightsquigarrow$ graph bounces

Work out the following: suppose $f(x) = x^3 - 6x^2 + 9x$.

1. What is its end behavior? [when $x \to \infty$, $f(x) \to ?$, when $x \to -\infty$, $f(x) \to ?$] $f(x) \to \infty$ when $x \to \infty$, $f(x) \to -\infty$ when $x \to -\infty$
2. Factor $f(x)$. $f(x) = x(x-3)^2$
3. What are its zeros and their multiplicity? 0 of multiplicity 1, 3 of multiplicity 2
4. What are its x-intercepts? $(0,0)$ and $(3,0)$
5. On which interval is $f(x) < 0$? $(-\infty, 0)$

Rational Functions

Idea: work with fractions of polynomials!

A *rational function* is the quotient of two polynomial functions

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$$

- No denominator can ever be 0: $\not\frac{4}{0}$ not defined.

So the *domain* of a rational expression are those x's for which $q(x) \neq 0$!

To simplify a rational expression, we factor both numerator and denominator, and then we cancel common factors

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b} \left(= \frac{a \cdot \not{c}}{b \cdot \not{c}} \right)$$

Vertical Asymptotes

Idea: behavior of rational functions around the inputs where they are NOT defined gives vertical asymptotes.

Finding Vertical Asymptotes for $f(x) = \dfrac{p(x)}{q(x)}$

1. simplify, if possible;
2. the line $\boxed{x = c}$ is a vertical asymptote, if 'c' a zero/root of the denominator;
3. the graph of $f(x)$ will never touch a vertical asymptote.

Work out the following: suppose $f(x) = \dfrac{2x-8}{x^2-16}$ and $g(x) = \dfrac{3x+2}{x-1}$.

1. f's domain?
   $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$
2. f's vertical asymptotes?
   $x = -4$
3. g's vertical asymptotes?
   $x = -1$
4. $g(x) \to ?$ when $x \to 1^-$?
   $g(x) \to -\infty$

End behavior and horizontal asymptotes

Idea: last time, behavior around undefined inputs;
now, behavior close to infinity!

Horizontal asymptotes

A *horizontal asymptote* exists at $y = a$ if the function values approach a as the input increases or decreases without bound $[f(x) \to a$ when $x \to \pm\infty]$.

- Similarly to polynomials, now reason with the ratio of leading terms.

Suppose we have the rational function $f(x) = \dfrac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$.

- If $n < m$, then $\boxed{y = 0}$ is a horizontal asymptote *larger denom*
- If $n = m$, then $\boxed{y = \frac{a_n}{b_m}}$ is a horizontal asymptote *same degree*
- If $n > m$, there is no horizontal asymptote *larger num*

$\star$ Write $\lim\limits_{x \to \pm\infty} f(x)$ for the end behavior of a function.

$\star$ The graph of a rational function may, or may not, cross horizontal asymptotes. It never crosses vertical asymptotes.

Work out the following: let $f(x) = \dfrac{3x-5}{2x+7}$ and $g(x) = \dfrac{2x}{x^3 - 6x^2 + 5x}$.

1. What is the horizontal asymptote of $f(x)$, if it exists? $y = \frac{3}{2}$
2. What is the horizontal asymptote of $g(x)$, if it exists? $y = 0$
3. What is the domain of $g(x)$ in interval notation? $D_g = (-\infty, 0) \cup (0, 1) \cup (1, 5) \cup (5, +\infty)$
4. What are the vertical asymptote(s) of $g(x)$, if they exist? $x = 1$, $x = 5$
5. $\lim\limits_{x \to 1^-} g(x) = ?$ $+\infty$

Graph Rational Functions

Idea: use asymptotes, intercepts and sign to graph rational functions!

Graph $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$

- Find the horizontal asymptote (ratio of leading terms / relative degrees give end behavior)
- Find the domain (need to only factor denominator $q(x)$).
- Simplify (also factor numerator $p(x)$); find the vertical asymptote(s).
- Solve $f(x) = 0$ to find its zeros x_i; x-intercepts are $(x_i, 0)$.
- y-intercept is $(0, f(0))$.
- [Sign Table] use zeros & domain to divide real line into intervals; determine sign in each by plugging test inputs in the simplified form.

Work out the following: let $f(x) = \dfrac{-x^2 - 3x + 4}{x^2 + 6x + 8}$.

1. What is its horizontal asymptote? $y = -1$
2. What is its domain? $D = (-\infty, -4) \cup (-4, -2) \cup (-2, +\infty)$
3. What are its vertical asymptotes? $x = -2$
4. What are its x-intercept(s)? What is its y-intercept? $(1, 0)$, $(0, \frac{1}{2})$
5. At which intervals is $f(x) > 0$? [sign table] $(-2, 1)$
6. (Sketch the graph) $\lim_{x \to -2^+} f(x)$? $+\infty$

Limits and Continuity

Idea: no matter if f is defined on c, can talk about f's behavior *close* to it.

$$\lim_{x\to c^+} f(x) = \text{behavior of } f(x) \text{ when } x \text{ approaches } c \text{ from the right}$$
$$\lim_{x\to c^-} f(x) = \text{behavior of } f(x) \text{ when } x \text{ approaches } c \text{ from the left}$$

Limit

If the left-sided and right-sided limits when x approaches c are the same number (not $\pm\infty$), then that equal value is called the *limit*

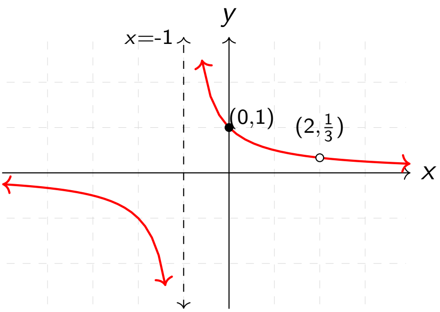
$$\lim_{x\to c} f(x)$$

of $f(x)$ when $x\to c$. If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$ or $\pm\infty$, then $\lim_{x\to c} f(x)$ DNE.

- A function $f(x)$ is *continuous* on c when $\lim_{x\to c} f(x)$ exists, and coincides with $f(c)$.

⋆ Intuitively, when we can draw the graph 'without picking up our pen'.

Work out the following: consider the function $f(x) = \dfrac{x-2}{(x+1)(x-2)}$



1. $\lim_{x \to -1^-} f(x)$? $\lim_{x \to -1^+} f(x)$? $\lim_{x \to -1} f(x)$? Is $f(x)$ continuous on -1? $-\infty, +\infty$, DNE, no
2. $\lim_{x \to 0^-} f(x)$? $\lim_{x \to 0^+} f(x)$? $\lim_{x \to 0} f(x)$? Is $f(x)$ continuous on 0? $1, 1, 1,$ yes
3. $\lim_{x \to 2^-} f(x)$? $\lim_{x \to 2^+} f(x)$? $\lim_{x \to 2} f(x)$? Is $f(x)$ continuous on 2? $\frac{1}{3}, \frac{1}{3}, \frac{1}{3},$ no

Co-variation of Numerator and Denominator

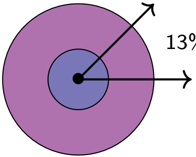
Idea: graph $f(x) = \dfrac{n(x)}{d(x)}$ directly from graphs of $n(x)$ and $d(x)$.

- Zeros of $d(x) \rightsquigarrow$ domain of $f(x)$
- Zeros of $n(x)$ [NOT common with $d(x)$] $\rightsquigarrow$ x-intercepts of $f(x)$
- Zeros of $d(x)$ [NOT common with $n(x)$] $\rightsquigarrow$ vertical asymptotes
- Common zeros of $d(x)$ and $n(x) \rightsquigarrow$ holes
- Ratio of leading terms of $n(x)$ and $d(x) \rightsquigarrow$ horizontal asymptote
- Ratio of y-intercepts of $n(x)$ and $d(x) \rightsquigarrow$ y-intercept of $f(x)$
- Signs and one-sided limits of $n(x)$ and $d(x) \rightsquigarrow$ rest of graph

$\star$ ...and vice versa, use graph of $f(x) = \frac{n(x)}{d(x)}$ to deduce info for $n(x)\&d(x)$!

Angle Measure

- An *angle* consists of two rays with a common endpoint, called its vertex. The *measure* of an angle is its amount of 'openness'!



⋆ Usual lengths of lines won't work; instead, take circle with vertex as center, and measure proportion of circle's circumference subtended by rays!

Degrees and Radians as units of angle measure $C = 360^\circ = 2\pi r$

- One *degree* 1° corresponds to $\frac{1}{360}$th of the circle's circumference, or $\frac{1}{360}$th of a complete rotation.
- One *radian* $1rad$ corresponds to an arc length equal to the radius of the circle, or $\frac{1}{2\pi}$ of a complete rotation.

⋆ Every circle's circumference is $2\pi \cong 6.28$ radius long, i.e. 6.28 radians!

Idea: all units measure percentage of circumference!

Angle Measure Conversion Formulas

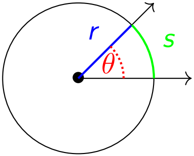
If β is the angle measure in degrees and θ in radians, then

$$\frac{\beta}{360} = \frac{\theta}{2\pi}$$

so $\beta = \frac{\theta}{2\pi}360^\circ$ and $\theta = \frac{\beta}{360}2\pi$ rad.

- The corresponding arc length can also be measured accordingly!

If θ is the angle in radians, r is the radius and s the arc length



then $\boxed{s = r\theta}$ in the same unit of length for s and θ.

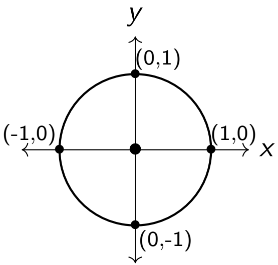
Work out the following:

1. What percentage of a circle's circumference does an angle of 4.5 rad correspond to (rounded to integer)? 72%
2. What is the measure in degrees of an angle that subtends to 60% of a circle's circumference? 216°
3. What is the measure in degrees of an angle 6 radians? What is the measure in radians of an angle 40°? (rounded) 344°, 0.7rad
4. What is the arc length of an angle 2.5 rad centered at a circle of 6 inches radius? 15 inches
5. For the same circle, what is the radian measure of an angle that subtends to an arc of length 36.6 inches? 6.1 radians

Circular Motion

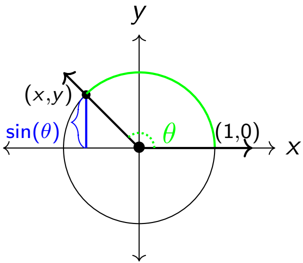
Idea: position the circle (with center the vertex of an angle) on a plane and talk about changes in x and y in various positions!

The *unit circle* is centered at $(0, 0)$ with radius one 'unit'

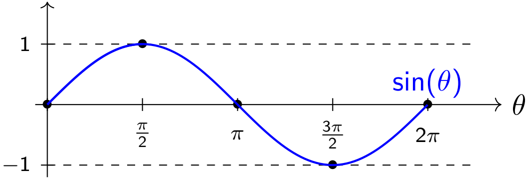


- $(1, 0)$ represents a horizontal distance x of 1 radius length.

The Sine Function

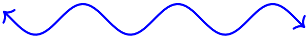


- The relationship between the angle measure θ rad and the y-coordinate of terminal point on unit circle is given by the *sine* function $f(\theta) = \sin(\theta)$

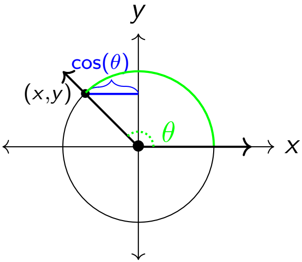


-1≤sin(θ)≤1

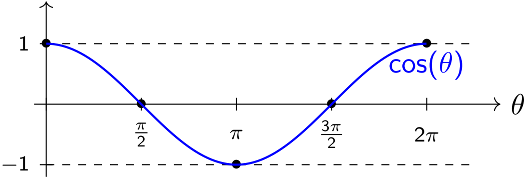
⋆ Its graph is *periodic*, i.e. repeats itself forever!



The Cosine Function



- The relationship between the angle measure θ rad and the x-coordinate of terminal point on unit circle is given by the *cosine function*



-1≤cos(θ)≤1

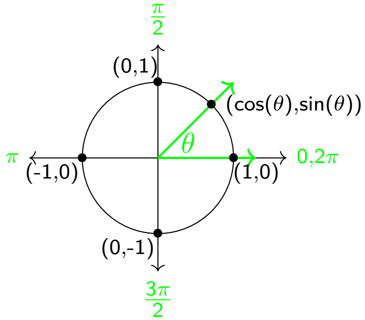
$\star$ Going around unit circle, know when sin(θ) and cos(θ) are $\lesseqgtr 0$!

Work out the following:

1. $\sin(0) =?$ $\cos(\pi) =?$ $\sin(\frac{3\pi}{2})$? $0, -1, -1$
2. If $\sin(\theta) > 0$ and $\cos(\theta) < 0$, at which quadrant is θ located? Second
3. Suppose an angle has $\sin(\theta) = -0.707$ and $\cos(\theta) = 0.707$. If the radius of the unit circle is 3 inches, what are the coordinates of the terminal point in inches? $(1.212in, -1.212in)$
4. What is the sine and cosine of an angle whose terminal point is $(-1cm, -1.732cm)$ on a circle of radius 2 cm (in radius lengths)? $\sin(\theta) = -0.866$, $\cos(\theta) = -0.5$
5. What is the radian measure of an angle 100°? 1.7453 radians

Sine and Cosine for circular motion

Idea: sine and cosine are always in terms of 'radius lengths', corresponding to angles in radians; that's how $\boxed{-1 \leq \sin(\theta), \cos(\theta) \leq 1}$ for any angle!



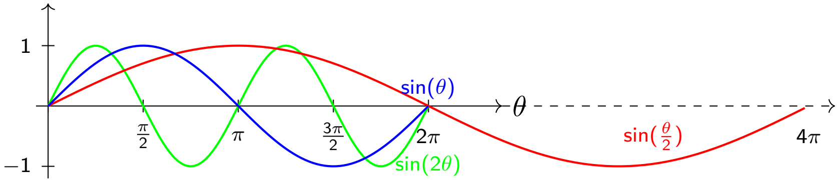
⋆ The actual distances they measure, if r is the length of the radius in some unit, can be found as $r \cdot \cos(\theta)$ and $r \cdot \sin(\theta)$ and vice versa!

Periodic functions

A *periodic function* is a function $f(x)$ with repeating output values after some regular interval of inputs: $f(x) = f(x + T)$, where T is the *period*.

- $\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$, so $T = 2\pi$.

⋆ What about sines and cosines of more complicated expressions of θ?



The period of $\sin(g(\theta))$ is given from solving $0 \leq g(\theta) \leq 2\pi$ for θ.

- The *amplitude* says how 'tall' the curve is: it is 1 in all cases above.

Work out the following:

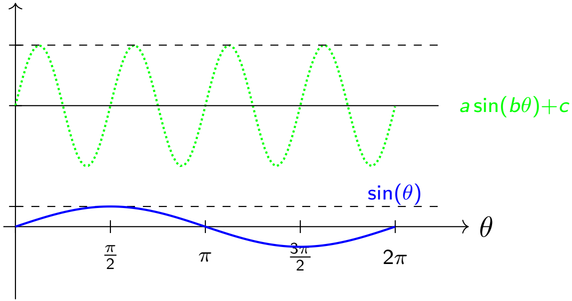
1. If $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, what is $\sin\left(-\frac{\pi}{4}\right)$? What is $\sin\left(\frac{3\pi}{4}\right)$? What is $\sin\left(\frac{9\pi}{4}\right)$? $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$
2. What is the period of $\cos\left(\frac{5\theta - 3}{2}\right)$? $T = 4\pi/5$
3. If a circular track has radius 90 feet, define functions $f(\theta)$ and $g(\theta)$ that express the horizontal and vertical distance from the center, in feet. $f(\theta) = 90\cos(\theta)$ ft, $g(\theta) = 90\sin(\theta)$ ft
4. What is the vertical distance for $\theta = \frac{\pi}{4}$, in feet? $45\sqrt{2}$ft

Transformations of Sine and Cosine

Idea: build up and graph trigonometric functions

Angular speed is the rate at which an object changes its angle θ (radians) per time t. Starting at $\theta=0$ with rate b rad per time unit, $\boxed{\theta(t) = bt}$.

- To graph $f(t) = a\sin(bt) + c$, use transformations: a relates to amplitude, bt relates to period, c shifts up or down.



Amplitude of periodic functions

Idea: find axis in the middle of graph, determine distance from max/min!

- The *midline* of a periodic function is the horizontal line located halfway between the maximum and minimum output values.
- The *amplitude* of a periodic function is the distance between the midline and the maximum or minimum outputs.

For trigonometric functions $a\sin(\theta)$ or $a\cos(\theta)$, amplitude is $|a|$.

Work out the following: suppose the radius of a circular trail is 3 meters.

1. If a bike covers 2.5 rotations per minute, what is its angular speed (in rad) per minute? 5π rad/min
2. What is the bike's angle at 3 minutes? What is its distance north of center, in meters? 15π rad, 0 m
3. Build a function $f(t)$ that determines vertical distance from center (in meters) in terms of time. $f(t) = 3\sin(5\pi t)$
4. What is its period? What is its amplitude? 0.4, 3

Graph Trigonometric Functions

Idea: accurately graph any function built from sine and cosine!

- Shifts and vertical stretches do not affect the period.
- For horizontal shift, need to factor the argument first.
- Vertical shift determines the midline of the graph.

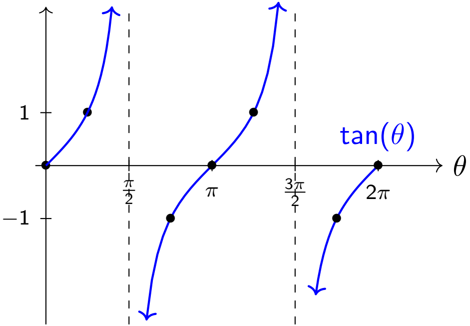
Methodology for graphing

1. Period (in graph, divide in four subintervals)
2. Amplitude (in graph, distance from midline to max/min)
3. Horizontal shift (in graph, shift left or right)
4. Vertical shift (in graph, shift up or down)

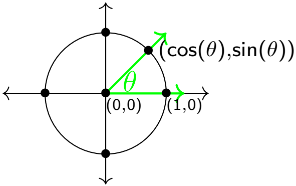
⋆ Always label the guiding plot points!

The Tangent function

Idea: the tangent is given by the slope of the terminal ray of an angle!

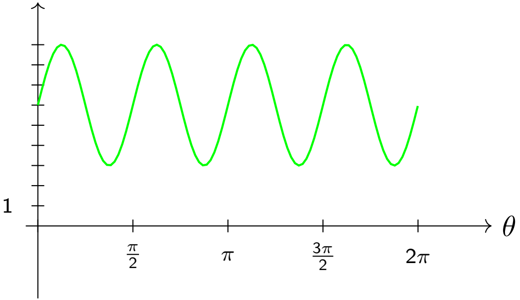


⋆ Recall: slope for line between (x_1, y_1) and (x_2, y_2) is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$!



$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Work out the following: consider the following graph



1. What is its period? Its midline? $T = \dfrac{\pi}{2}$, midline is $y = 6$
2. Max and min values? Amplitude? max value is 9, min value is 3, amplitude is 3
3. What is a possible function for the graph? $f(\theta) = 3\sin(4\theta) + 6$
4. Biker starts at 9 o'clock position on circular trail of radius 10ft, CCW. Function that expresses her horizontal distance in feet? $f(\theta) = 10\cos(\theta + \pi)$

Inverse Trigonometric Functions

Idea: study inverses f^{-1} of basic trig functions: they reverse f's process!

Recall:

- the domain of f^{-1} is the range of f and vice versa
- f^{-1} is the unique function such that $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$

$\star$ Infinitely many angles with same sine? Restrict to some that give whole spectrum of vales in $[-1, 1]$ without repetition (where it is one-to-one...)

- The *inverse sine*, or *arcsine*, has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\sin(\theta) = x \Leftrightarrow \theta = \arcsin(x)$$

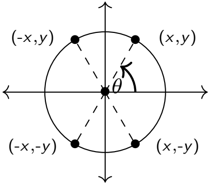
- The *inverse cosine*, or *arccosine*, has domain $[-1, 1]$ and range $[0, \pi]$

$$\cos(\theta) = x \Leftrightarrow \theta = \arccos(x)$$

- The *inverse tangent*, or *arctangent*, has domain $(-\infty, \infty)$&range $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\tan(\theta) = x \Leftrightarrow \theta = \arctan(x)$$

- Based on the angle's quadrant, using reflections can deduce all points on unit circle with required trigonometric value.



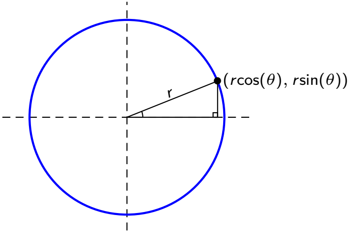
Work out the following:

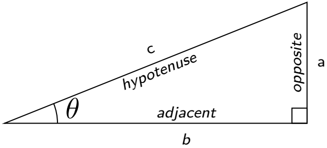
1. $\tan(\pi)$? $\tan(\frac{3\pi}{2})$? $\tan(\frac{5\pi}{4})$? 0, DNE, 1
2. For which quadrant(s) is the tangent negative? 2nd, 4th
3. For which θ on the third quadrant do we have that $\sin(\theta) = -\frac{1}{2}$? 3.6635 radians

Right Triangle Trigonometry

Idea: express sine, cosine and tangent using the sides of a right triange!

Every right triangle can be placed inside a circle with r=hypotenuse:



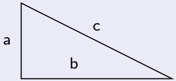


$$\sin(\theta) = \frac{a}{c}$$
$$\cos(\theta) = \frac{b}{c}$$
$$\tan(\theta) = \frac{a}{b}$$

⋆ For any right triangle, knowing any two sides we can find the third!

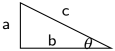
The Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse:



$$\boxed{a^2 + b^2 = c^2}$$

Work out the following:

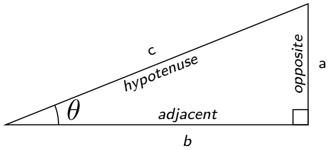
1. Solve $2\tan(\theta) - 16 = 0$, for $\theta \in [0, 2\pi]$. (use calculator) 1.4464, 4.5864 radians
2. What is $\arcsin(\sin(\frac{\pi}{2}))$? (do not use calculator) $\frac{\pi}{2}$!
3. Suppose $\sin(\theta) = \frac{4}{5}$. What is $\cos(\theta)$ and $\tan(\theta)$? (without calculator; sketch one such right triangle) $\cos(\theta) = \frac{3}{5}$, $\tan(\theta) = \frac{4}{3}$
4. 

If $c = 26$cm & $\cos(\theta) = 0.5$, what is a and b? $a \approx$22.5cm, $b = 13$cm

Applications of Right Triangle Trigonometry

Idea: use sine, cosine and tangent functions to tackle various real-life problems including right triangles!

- Always start by drawing a picture that represents the problem
- Place the known quantities on it; give names to the unknown ones
- Decide which trigonometric function matches the data
- Model by functions? Identify what should be the input and output and solve accordingly



$$\sin(\theta) = \frac{a}{c} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{c}{a}$$

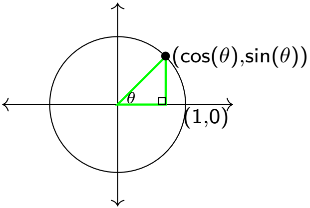
$$\cos(\theta) = \frac{b}{c} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{c}{b}$$

$$\tan(\theta) = \frac{a}{b} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{b}{a}$$

⋆ The functions *cosecant*, *secant* and *cotangent* are the reciprocal ones!

Trigonometric Identities

Idea: show that trig functions are related via useful equalities, for any θ!



Pythagorean Identity

$\sin^2(\theta) + \cos^2(\theta) = 1$

Work out the following:

1. If the Eiffel Tower is 1063 feet, what is the distance when a tourist whose angle is 80°? Write a function $d(\theta)$ (in feet) that expresses a tourist's ground distance in terms of the angle. 187.5ft, $d(\theta) = \frac{1063}{\tan(\theta)}$
2. A boat leaves at angle 20° with speed 10 mph. What is the distance covered in 3 hours, and what is its location north? 30mi, 10.26mi N
3. (same boat) write functions $f(t)$ and $g(t)$ of time for distance and north location. $f(t) = 10t$, $g(t) = 10\sin(20)t$
4. Simplify $\sqrt{4\sin^2(\theta) + 4\cos^2(\theta)}$. 2

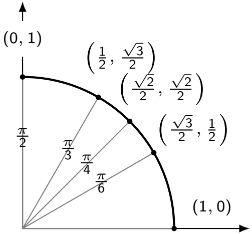
Trigonometric Identities

Idea: algebraically or geometrically deduce many trigonometric relations.

Important Trigonometric Identities

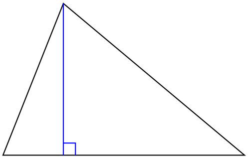
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$
- $\tan^2(x) + 1 = \sec^2(x)$
- $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$

$\star$ Line-by-line: *Pythagorean*, *Sum*, *Double Angle*, *Power Reducing*.



Trigonometry in Non-Right Triangles

Idea: in an arbitrary triangle, can draw any of the three *altitudes* and split it up it two right triangles!



- Sum of angles of any triangle is 180°: two angles always give the third.
- The area of a triangle is $\frac{1}{2} \cdot b \cdot h$, where h is the length of any altitude and b is the length of the targeting side.

Work out the following:

1. $\sin(\frac{\pi}{4})$? $\cos(\frac{\pi}{3})$? $\sin(\frac{7\pi}{6})$? $\frac{\sqrt{2}}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$
2. $(7\sin^2(\theta) + 7\cos^2(\theta))^2$? 49
3. $\dfrac{12}{6\sec^2(x) - 6\tan^2(x)}$? 2
4. Solve $4\sin(2\theta) - 2\sqrt{2} = 0$ on $[0, 2\pi]$. $\frac{\pi}{8}$, $\frac{3\pi}{8}$